

Christopher C. Tisdell

Partial Differential Equations

YouTube Workbook with Dr Chris Tisdell

CHRISTOPHER C. TISDELL

PARTIAL DIFFERENTIAL EQUATIONS

YOUTUBE WORKBOOK
WITH DR CHRIS TISDELL

Partial Differential Equations: YouTube Workbook with Dr Chris Tisdell

1st edition

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How to use this book

Hi and thanks so much for choosing to download this book. You're now part of a learning community of over 10 million readers around the world who engage with my learning resources – welcome!

This workbook is designed to be used in conjunction with the my free online video tutorials. The subsections in each chapter have their own dedicated video tutorial.

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While watching each video, fill in the spaces provided after each example in the workbook and annotate to the associated text.

You can also access the above via my YouTube channel

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There has been an explosion in books that connect text with video since the pioneering work *Engineering Mathematics: YouTube Workbook* [51].

I really hope that you will find this book to be useful. I'm always keen to get feedback from you about how to improve my books and your learning process.

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What makes this book different?

A search involving the term “partial differential equations” on Amazon.com returns over 5,000 results for books. Does the world really need another textbook in this area?

So, what makes this book different? Some points of distinction between this book and others include:

- Open learning design
- Multimodal learning format
- Active learning spaces.

What do I mean by these terms?

Open learning design

My tagline of “everyone deserves access to learning on a level playing field” draws on the belief that open access to education is a public right and a public good. The design of this book follows these beliefs in the sense that the book is absolutely free; and does not require any special software to function. The book can be printed out or used purely in electronic form.



Multimodal learning format

In recent years, graphics have played a more common role within textbooks, especially with the move away from black and white texts to full colour. However, the traditional textbook is still percieved as being static and unimodal in the sense that you can read the text. While some texts have attempted to use video as an “added extra”, the current textbook aims to fully integrate online video into the learning experience. When the rich and expressive format of video is integrated with simple text it leads to what I call a multimodal learning experience (sight, sound, movement etc), going beyond what a traditional textbook can offer.

Active Learning Spaces

It’s far too easy to sit back and just passively “watch” – whether it’s a lecture, a tutorial, a TV show or an online video. However, learning is not a spectator sport. I believe in the power of active learning: that is, in learners doing things and thinking about what they are doing and what they have done.

To encourage active learning, this book features blank spaces where learners are required to actively engage by taking notes, making annotations, drawing diagrams and the like. I call these blank spaces “active learning spaces”.

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Chapter 1

The Transport Equation

1.1 Introduction - Where are we going?

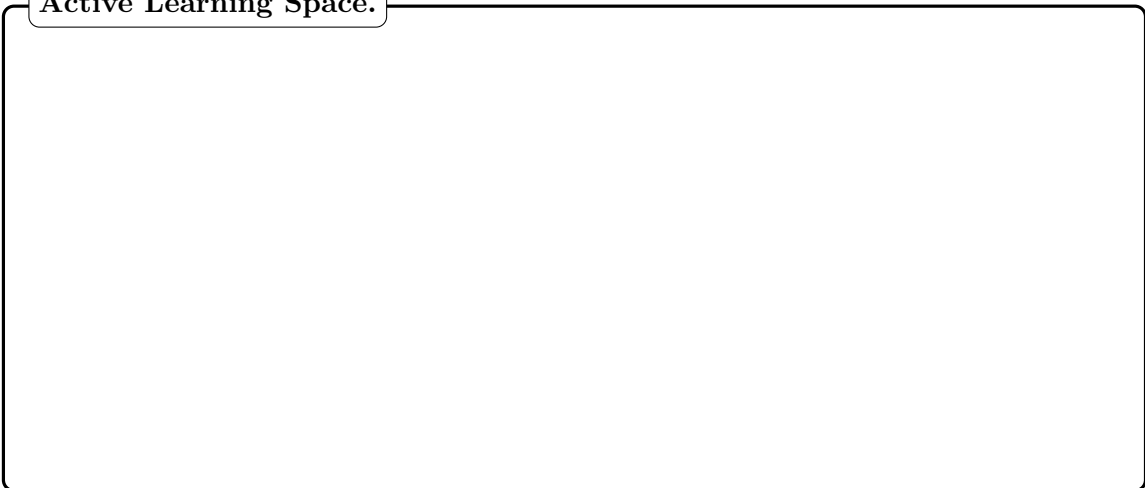
View this lesson on YouTube [1]

- We will derive and solve some basic forms of a famous partial differential equation (PDE) known as **the transport equation**.
- The discussion on the derivation of the transport equation will shed light on how partial differential equations arise generally.
- The solution methods for the transport equation will hint at how more general PDE might be attacked and solved.

Here are some forms of the transport equation in increasing order of complexity:

- $u_t + cu_x = 0$, where $u = u(x, t)$ and c is a constant;
- $\rho_t + q_x = 0$, where ρ and q depend on x and t ;
- $u_t + \vec{c} \cdot \nabla u = 0$, where $u = u(x, y, z, t)$ and \vec{c} is a constant vector.
- $u_t + \vec{c} \cdot \nabla u = f(x, y, z, t)$.

Active Learning Space.



1.2 Solution Method to Transport Equation via Directional Derivatives

Consider the transport equation

$$u_t + cu_x = 0, \quad \text{where } c = \text{constant.}$$

- We look for functions $u = u(x, t)$ that satisfy the transport equation via directional derivatives.
- Remember from vector calculus that the directional derivative of u in the direction of a vector \vec{v} is

$$D_{\hat{v}}u = \hat{v} \bullet \nabla u$$

where \hat{v} is the unit vector in direction of \vec{v} and $\nabla u = (u_x, u_t)$.

- In particular, if $D_{\hat{v}}u = 0$ at all points then u is constant along all lines that are tangential to \vec{v} .

Rewrite the transport equation as the dot product form

$$0 = u_t + cu_x = (c, 1) \bullet (u_x, u_t)$$

which is equivalent to

$$D_{(c,1)}u = 0.$$

- Hence every solution u must be constant along straight lines that are tangential to $(c, 1)$. In this case, the lines are also parallel to the vector $(c, 1)$.
- Now, lines parallel to vector $(c, 1)$ have slope $dt/dx = 1/c$. Let's write this as $dx/dt = c/1$ and so their equation is

$$x = ct + K, \quad \text{where } K = \text{constant}$$

and rearrange to form $x - ct = K$. This equation describes an infinity of lines tangential (and parallel) to the vector $(c, 1)$.

- Along each of these lines u is constant. That is, u depends only on $x - ct$ and so we have the form

$$u(x, t) = f(x - ct)$$

for arbitrary f . To see this, take a particular value of K , say K_1 and consider the line $x - ct = K_1$. Along this line $u = \text{constant}$, call this constant $f(K_1)$. Hence

$$u = f(K_1) = f(x - ct).$$

View this lesson on YouTube [2]

Example.

Solve:

a) $u_t + 2u_x = 0$;

b) $2u_t - u_x = 0$;

c) $u_t + 5u_x = 0$, $u(x, 0) = e^x$.

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1.3 Derivation of the Most Basic Transport Equation

View this lesson on YouTube [3]

We now show how to derive the simple transport equation

$$u_t + cu_x = 0.$$

Consider water flowing through a thin pipe with fixed cross-section.

- The water moves in one direction only with a constant speed c (unit length / unit time).
- We take the direction of flow of water to be the positive x direction.
- Suppose that suspended within the water is salt (or any other pollutant).
- No water or salt enters or exits the pipe (conservation property).

Let $u = u(x, t)$ denote the concentration (or density, in unit mass / unit length, say grams / cm) of salt present at position x and time t .



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Consider an interval $[a, b]$ on the x -axis. Let us derive the total mass of salt (say, in grams) in $[a, b]$ at a fixed time value t_0 .

Total mass of salt in $[a, b]$ at time t_0

$$\approx \sum \Delta M_i \approx \sum u(x_i^*, t_0) \Delta x \rightarrow \int_a^b u(x, t_0) dx.$$

Now, at time $t_0 + h$ the same molecules have moved to the right by a distance of ch units (cm). This is to be expected, as the water moves with constant speed c . Hence

$$\int_a^b u(x, t_0) dx = \int_{a+ch}^{b+ch} u(x, t_0 + h) dx.$$

Differentiate the above equation with respect to b by applying the fundamental theorem of calculus to form

$$u(b, t_0) = u(b + ch, t_0 + h).$$

Now differentiate both sides of the previous equation with respect to h via the chain rule, to obtain

$$0 = cu_x(b + ch, t_0 + h) + u_t(b + ch, t_0 + h)$$

(just let $x = b + ch$ and let $t = t_0 + h$). Finally, let $h = 0$ to form

$$0 = cu_x(b, t_0) + u_t(b, t_0).$$

Now, b and t_0 are totally arbitrary, so this is just

$$u_t + cu_x = 0.$$

Thus we have derived our PDE. □

The above simple transport equation says that the rate of change u_t of concentration is proportional to the (one-dimensional) gradient of u .

View this lesson on YouTube [4]

Example.

Solve:

$$u_t + 2xu_x = 0.$$

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View this lesson on YouTube [5]

Example.

Solve:

$$u_t + u_x + 7u = 0, \quad u(x, 0) = e^x.$$

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1.4 Transport Equation Derivation

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Consider a one-dimensional road with cars moving along it in one direction only. Let $\rho(x, t)$ denote the concentration (or density) of cars at position x and time t (in mass / unit of length).

The total mass of cars at time t within an arbitrary segment of the road $[a, b]$ is

$$M(t) = \int_a^b \rho(x, t) dx.$$

Now, at time t , the flux (or flow rate) Q of cars into the interval $[a, b]$ (in mass / unit time) can be written in terms of the following difference:

$$\begin{aligned} Q = & \\ & \text{(rate of cars entering the region at } x = a) \\ & - \text{(rate of cars leaving region at } x = b). \end{aligned}$$

Since no cars are created or destroyed on the road, the conservation of mass law yields rate of change of mass in $[a, b] =$ flux into region $[a, b]$,

that is

$$\frac{dM}{dt} = Q = q(a, t) - q(b, t).$$

Now, by the fundamental theorem of calculus

$$\begin{aligned} Q &= q(a, t) - q(b, t) \\ &= -q(x, t) \Big|_{x=a}^{x=b} \\ &= - \int_a^b q_x(x, t) dx. \end{aligned}$$

Thus

$$\frac{dM}{dt} = - \int_a^b q_x(x, t) dx.$$

Also,

$$\begin{aligned} \frac{dM}{dt} &= \frac{d}{dt} \int_a^b \rho(x, t) dx \\ &= \int_a^b \frac{\partial}{\partial t} \rho(x, t) dx \\ &= \int_a^b \rho_t(x, t) dx \end{aligned}$$

where we have applied Leibniz' rule to differentiate under the integral sign.

Equating our expressions for dM/dt we obtain

$$\int_a^b \rho_t(x, t) dx = - \int_a^b q_x(x, t) dx$$

ie

$$\int_a^b [\rho_t(x, t) + q_x(x, t)] dx = 0$$

and since this is true for every interval $[a, b]$ the integrand must be identically zero, ie

$$\rho_t(x, t) + q_x(x, t) = 0.$$

□

If $q = c\rho$ ($c = \text{const.}$) then we obtain the familiar PDE

$$\rho_t + c\rho_x = 0.$$



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Example.

(Challenge Problem) Show that the nonhomogeneous problem

$$u_t + cu_x = g(x, t) \quad (1.4.1)$$

$$u(x, 0) = h(x) \quad (1.4.2)$$

is equivalent to the integral equation

$$u(x, t) = h(x - ct) + \int_0^t g(x + (s - t)c, s) ds, \quad x \in \mathbb{R}, t \geq 0.$$

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Example.

(Independent Learning)

A general form of the Cauchy problem associated with the transport equation is

$$u_t + \vec{c} \bullet \nabla u = g(\vec{x}, t) \quad (1.4.3)$$

$$u(\vec{x}, 0) = h(\vec{x}) \quad (1.4.4)$$

where \vec{x} is an n -dimensional vector and \vec{c} is a constant vector. Show that (1.4.3), (1.4.4) is equivalent to the integral equation

$$u(\vec{x}, t) = h(\vec{x} - \vec{c}t) + \int_0^t g(\vec{x} + (s-t)\vec{c}, s) ds, \quad \vec{x} \in \mathbb{R}^n, t \geq 0.$$

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Chapter 2

Solve PDE via a Change of Variables

2.1 Change of Co-ordinates and PDE

View this lesson on YouTube [8]

Consider the PDE

$$au_x + bu_y + cu = f(x, y) \quad (2.1.1)$$

where a , b and c are constants. We make a change of co-ordinates, equivalent to rotating the axes, so that the new “horizontal” axis lies along the vector (a, b) . This will reduce the PDE to an ODE.

Important Idea.

(Change of coordinates)

From (2.1.1), let

$$\tan \alpha = b/a$$

and set

$$\xi = x \cos \alpha + y \sin \alpha$$

$$\eta = -x \sin \alpha + y \cos \alpha.$$

Example.

Solve

$$u_x + u_y - 2u = 0.$$

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2.2 Non-constant Co-efficients

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We now show how a change of variables can be applied to solve the more general PDE

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y). \quad (2.2.1)$$

Let us construct a new set of co-ordinates (ξ, η) variables (which may no longer be orthogonal, as in the constant co-efficient case) such that the PDE (2.2.1) reduces to an ODE.

How to determine $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ as a change of variables? Consider $u(\xi, \eta)$ and to recast (2.2.1) in terms of ξ and η we use the chain rule to compute

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y. \end{aligned}$$

Substitution into (2.2.1) yields

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta + cu = d.$$

Now, since we want the above to become an ODE, we require

$$a\xi_x + b\xi_y = 0, \quad \text{or} \quad a\eta_x + b\eta_y = 0.$$

Note that we just have two simple PDE, which we know how to solve. Suppose we want to solve the first PDE. We know that the solution ξ is constant along curves that satisfy

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}. \quad (2.2.2)$$

Now the general implicit form of solution to (2.2.2) is $f(x, y) = C$ and so we can choose $\xi(x, y) = f(x, y)$.

For the other variable, η , we may choose

$$\eta(x, y) = x.$$

These change of co-ordinates are non-degenerate, as the determinant of the Jacobian is non-zero

$$J := \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_y \neq 0.$$

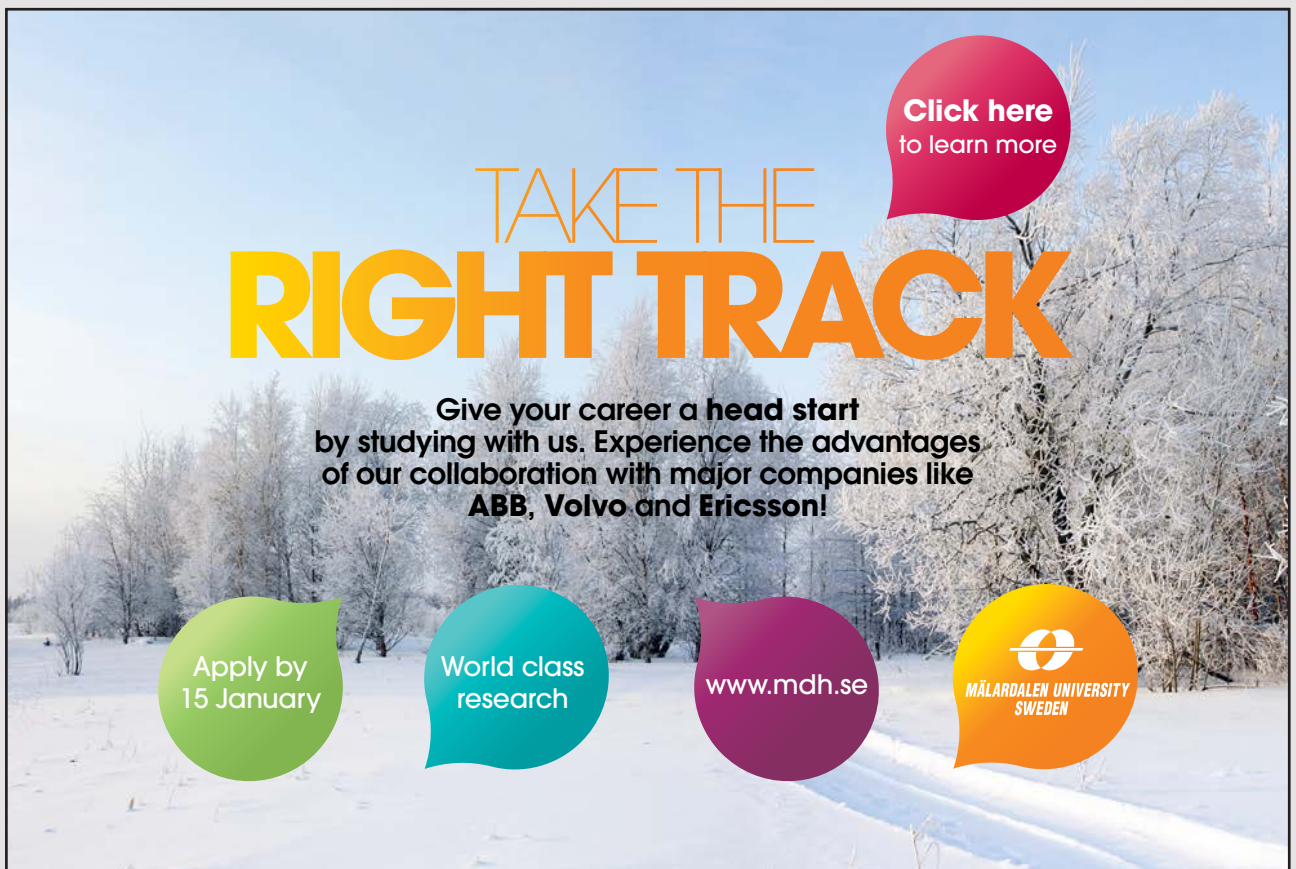
With these new variables, our PDE (2.2.1) becomes

$$a(\xi, \eta)u_\eta + u(\xi, \eta) = d(\xi, \eta).$$

which can be solved by standard linear ODE techniques.

If we solved $a\eta_x + b\eta_y = 0$ instead, then argument is similar. We would form η from (2.2.2) and then choose $\xi = x$. The PDE (2.2.1) becomes

$$a(\xi, \eta)u_\xi + u(\xi, \eta) = d(\xi, \eta).$$



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Example.

Solve

$$xu_x - yu_y + u = 2.$$

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Chapter 3

PDE and the method of characteristics

3.1 Introduction

View this lesson on YouTube [11]

Dr. Felix T. Smith at Stanford Research Institute:

“I’m afraid I don’t understand the method of characteristics.”



Prof John von Neumann (pictured):

“In mathematics you don’t understand things. You just get used to them.”

- We will introduce an approach to solve partial differential equations (PDE), known as the method of characteristics.
- The discussion relies on powerful geometrical arguments.
- The method enables us to solve a wide range of problems, including linear, semi-linear and quasi-linear PDE (which are defined as):

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y), \quad (\text{linear})$$

$$a(x, y)u_x + b(x, y)u_y = f(x, y, u), \quad (\text{semi-linear})$$

$$a(x, y, u)u_x + b(x, y, u)u_y = f(x, y, u), \quad (\text{quasi-linear}).$$

3.2 The semi-linear Cauchy problem

We shall consider the PDE

$$a(x, y)u_x + b(x, y)u_y = f(x, y, u) \quad (3.2.1)$$

which is first-order and termed “semi-linear”. It includes the linear form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = g(x, y)$$

with $f(x, y, u) = g(x, y) - c(x, y)u$. We shall couple our PDE with an initial condition

$$u(x, 0) = u_0(x) \quad (3.2.2)$$

or $u(0, y) = u_1(y)$. The pair (3.2.1), (3.2.2) is known as a **semi-linear Cauchy problem**. As we have seen in previous examples, the solution to a Cauchy problem can be pictured as a surface $z = z(x, y)$ lying in 3D space. We now show how visualization and geometry can shed light on the solution to

$$a(x, y)u_x + b(x, y)u_y = f(x, y, u), \quad u(x, 0) = u_0(x).$$



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Let us write the equation of the integral surface¹ as

$$F(x, y, u) := u(x, y) - u = 0.$$

Note that the first u is a function, while second u is a variable. For example, suppose $u(x, y) = x - y^2$ then $F = x - y^2 - u$. Now, the gradient from vector calculus enables us to calculate a normal vector \vec{n} to the surface $F = 0$ via

$$\vec{n} = \nabla F = u_x \vec{i} + u_y \vec{j} - \vec{k}$$

(which will happen to be downward-pointing). Now notice that we may rewrite our PDE as the dot product

$$(a(x, y), b(x, y), f(x, y, u)) \bullet (u_x(x, y), u_y(x, y), -1) = 0.$$

From above, we see that this is

$$(a(x, y), b(x, y), f(x, y, u)) \bullet \vec{n} = 0$$

so that the vector

$$\vec{v} := a(x, y)\vec{i} + b(x, y)\vec{j} + f(x, y, u)\vec{k}$$

is normal to \vec{n} . Now, since \vec{n} is normal to the surface $F = 0$, this means that \vec{v} must be **tangential** to the surface $F = 0$ and hence must lie in the tangent plane to $F = 0$ at every point. Thus, the PDE provides a geometric requirement that any integral surface through a given point be tangent to the vector

$$\vec{v} = a(x, y)\vec{i} + b(x, y)\vec{j} + f(x, y, u)\vec{k}.$$

That is, if we start at some point of the initial condition (which must lie in the integral surface) and move in the direction of the vector \vec{v} (which we know just from looking at the PDE) then we move along a curve that lies entirely within the surface $F = 0$.

Important Idea.

This curve is called a **characteristic curve**.

Now, we can describe points on a characteristic curve via a parametrization

$$\vec{p}(t) = x(t)\vec{i} + y(t)\vec{j} + u(t)\vec{k}.$$

If we differentiate \vec{p} with respect to t then we obtain a tangent vector

$$\vec{p}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + u'(t)\vec{k}.$$

Now, this tangent vector must belong to the tangent plane to the surface $F = 0$ at the given point.

Since the characteristic is obtained by moving in the direction of \vec{v} , we conclude that \vec{p}' and \vec{v} must be proportional, ie $\vec{p}' = \lambda\vec{v}$ for some λ , or

$$\frac{dx/dt}{a(x, y)} = \frac{dy/dt}{b(x, y)} = \frac{du/dt}{f(x, y, u)}$$

¹We call a surface $u = u(x, y)$ that is a solution to the Cauchy problem an **integral surface**. The idea behind the name is due to integrations being used to solve differential equations.

or just

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f}. \quad (3.2.3)$$

From (3.2.3) we can form two pairs of ODE

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}, \quad \frac{du}{dx} = \frac{f(x, y, u)}{a(x, y)}$$

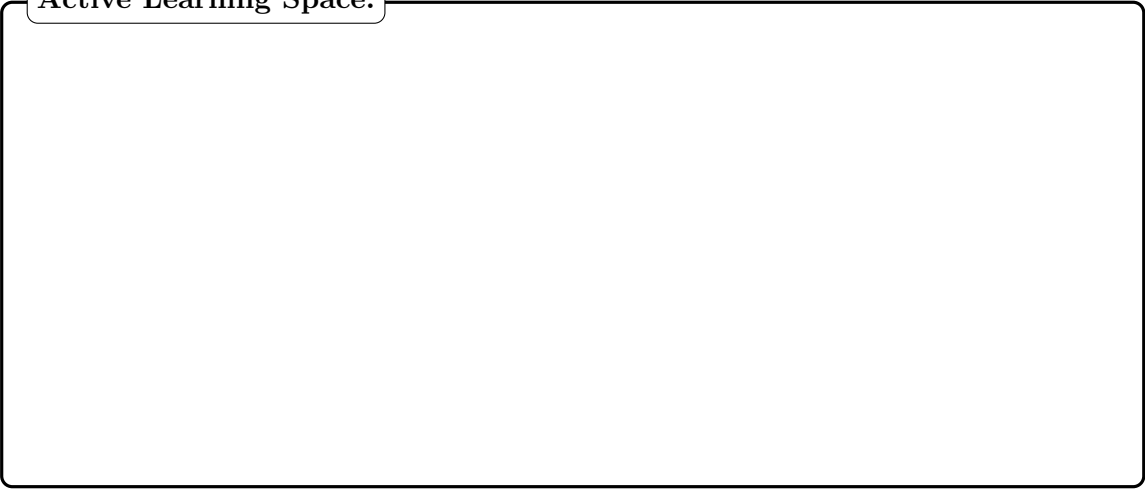
or

$$\frac{dx}{dy} = \frac{a(x, y)}{b(x, y)}, \quad \frac{du}{dy} = \frac{f(x, y, u)}{b(x, y)}.$$

The solutions to the above pairs of equations determine the characteristics of the PDE

Let's discuss some examples to see how these characteristics may be applied.

Active Learning Space.



Example.

Solve

$$2u_x + u_y = u, \quad u(x, 0) = \frac{x}{2} + e^x.$$

Active Learning Space.

To solve the semi-linear PDE

$$a(x, y)u_x + b(x, y)u_y = f(x, y, u)$$

via the method of characteristics:

- Consider the system:

$$\frac{dx}{a(x, y)} = \frac{dy}{b(x, y)} = \frac{du}{f(x, y, u)}.$$

- Solve (if possible) the following ODE for $h(x, y) = c_1$

$$\frac{dx}{a(x, y)} = \frac{a(x, y)}{b(x, y)} \left(\text{respectively } \frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}. \right)$$

- Solve for $j(x, y, u) = c_2$

$$\frac{du}{dy} = \frac{f(x, y, u)}{b(x, y)}, \left(\text{respectively } \frac{du}{dx} = \frac{f(x, y, u)}{a(x, y)}. \right)$$

- Write $j(x, y, u) = G(h(x, y))$ for arbitrary G .

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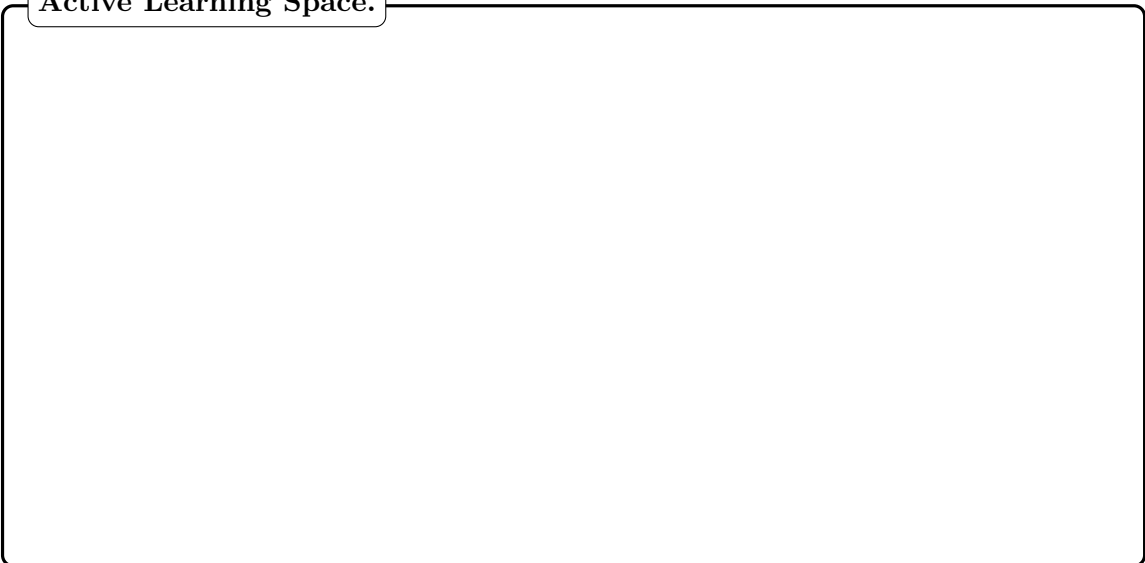
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Why $j = G(h)$??

- Remember that we are solving our ODEs along characteristic curves.
- To get the value of u we need to solve along every characteristic and each time we solve it we obtain undetermined constant c_2 .
- Now, c_2 can vary between characteristics, but must be the same along any particular characteristic.
- The particular characteristic curve that we're considering is determined by the constant c_1 .

Hence, there must be a functional relationship between c_2 and c_1 .

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Example.

Solve

$$tu_t + u_x = 0.$$

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3.3 Quasi-linear Case

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The quasi-linear PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = f(x, y, u) \quad (3.3.1)$$

can be treated in a similar manner to the semi linear case.

If we can determine two (linearly independent) first integrals of

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{f(x, y, u)}$$

of the form $h(x, y, u) = c_1$ and $j(x, y, u) = c_2$, then the general solution to the PDE (3.3.1) may be written in implicit form

$$j(x, y, u) = F(h(x, y, u))$$

where F is an arbitrary function.

Example.

Solve

$$uu_x + u_y = 0, \quad u(x, 0) = \phi(x).$$

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Example.

Solve

$$(y + u)u_x + yu_y = x - y.$$

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Example.

Solve

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u, \quad u(x, 3x) = x^2.$$

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Chapter 4

Solving the Wave Equation

4.1 Introduction

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- We will discover a number of approaches to solving a famous PDE known as the (one-dimensional) wave equation

$$u_{tt} = c^2 u_{xx}.$$

- The discussion ranges from: factorizations; to integral representations; to substitutions.
- The ideas will guide us on solving a range of other second-order problems.



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4.2 A Factoring Approach

Consider the PDE

$$u_{tt} = c^2 u_{xx}, \quad c = \text{constant}. \quad (4.2.1)$$

Notice that if we define the differential operator L by

$$L(u) := u_{tt} - c^2 u_{xx}$$

then (4.2.1) is just

$$L(u) = 0.$$

We can “factor” L with the aim of obtaining two first-order PDE in the following way. Observe that

$$L(u) = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u. \quad (4.2.2)$$

This means that we begin with a function $u(x, t)$, we compute $u_t + cu_x$. Let the result be v and then compute $v_t - cv_x$ and you should get zero. Thus, if we let

$$v = u_t + cu_x$$

then we have formed two first-order PDEs, namely

$$v_t - cv_x = 0 \quad (4.2.3)$$

and

$$u_t + cu_x = v. \quad (4.2.4)$$

Note that (4.2.3) is a transport-type equation, which has solution

$$v(x, t) = j(x + ct)$$

where $j(x) = u(x, 0)$ and so (4.2.4) becomes an inhomogeneous transport equation, namely

$$u_t + cu_x = j(x + ct). \quad (4.2.5)$$

This is just a nonhomogenous transport equation (which we know how to solve). The solution to (4.2.5) is

$$u(x, t) = h(x - ct) + \int_0^t j(x + (s - t)c + cs) ds$$

where $u(x, 0) = h(x)$. Now if $G' = j$ then we have

$$\begin{aligned} u(x, t) &= h(x - ct) + \int_0^t j(x + (2s - t)c) ds \\ &= h(x - ct) + \frac{1}{2c} [G(x + (2s - t)c)]_{s=0}^{s=t} \\ &= h(x - ct) + \frac{1}{2c} [G(x + ct) - G(x - ct)] \\ &= f(x + ct) + g(x - ct) \end{aligned}$$

where we have let $g := h - G/2c$ and $f := G/2c$ to make things a little prettier.

Important Idea.

The general solution (called D'Alembert's solution) to the wave equation $u_{tt} = c^2 u_{xx}$ is

$$u(x, t) = f(x + ct) + g(x - ct).$$

There are other approaches to solve the wave equation (4.2.1). For example, if we introduce a change of co-ordinates

$$\xi = x + ct, \quad \eta = x - ct$$

then apply the chain rule to form

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = \left(-2c\frac{\partial}{\partial \xi}\right)\left(2c\frac{\partial}{\partial \eta}\right)u = 0$$

which gives (since $c \neq 0$)

$$u_{\xi\eta} = 0.$$

Integrate this equation twice to form

$$\begin{aligned} u &= f(\xi) + g(\eta) \\ &= f(x + ct) + g(x - ct) \end{aligned}$$

which is identical to the form of u we obtained earlier. The previous discussion indicates that there are **two** families of characteristic curves: the lines

$$x + ct = K, \quad x - ct = N.$$

The most general solution to the wave equation (4.10.1) is the sum of two functions:

- $f(x + ct)$ represents a wave moving to the left at speed c ;
- $g(x - ct)$ is a wave moving to the right with speed c .

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4.3 Initial Value Problem: Wave Equation

Consider the following IVP

$$u_{tt} = c^2 u_{xx} \tag{4.3.1}$$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \tag{4.3.2}$$

We know that the general solution to $u_{tt} = c^2 u_{xx}$ is

$$u(x, t) = f(x + ct) + g(x - ct)$$

so the initial conditions become

$$\begin{aligned} \phi(x) &= f(x) + g(x) \\ \psi(x) &= cf'(x) - cg'(x). \end{aligned}$$

We wish to solve for f and g (in terms of ϕ and ψ). Let's differentiate the first equation to form

$$\begin{aligned} \phi'(s) &= f'(s) + g'(s) \\ \psi(s) &= c[f'(s) - g'(s)]. \end{aligned}$$

where we have moved from x to s for a little convenience.

The two equations may be combined in the following forms

$$\begin{aligned} \phi' &= f' + g' \\ \frac{\psi}{c} &= f' - g'. \end{aligned}$$

which leads to

$$\begin{aligned} f' &= \frac{1}{2} \left(\phi' + \frac{\psi}{c} \right) \\ g' &= \frac{1}{2} \left(\phi' - \frac{\psi}{c} \right). \end{aligned}$$

Integration of each equation then yields

$$\begin{aligned} f(s) &= \frac{1}{2} \left(\phi(s) + \int_0^s \frac{\psi}{c} dz \right) + A \\ g(s) &= \frac{1}{2} \left(\phi(s) - \int_0^s \frac{\psi}{c} dz \right) + B. \end{aligned}$$

Now, $\phi(x) = f(x) + g(x)$ gives $A + B = 0$. Now, substituting $s = x + ct$ into the formula for f and substituting $s = x - ct$ into the formula for g we obtain

$$\begin{aligned} u(x, t) &= \frac{1}{2} \phi(x + ct) + \int_0^{x+ct} \frac{\psi(z)}{2c} dz + \frac{1}{2} \phi(x - ct) - \int_0^{x-ct} \frac{\psi(z)}{2c} dz \\ &= \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds. \end{aligned}$$

4.4 Inhomogenous Case

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Consider the inhomogeneous problem associated with the wave equation

$$u_{tt} = c^2 u_{xx} + h(x, t), \quad t > 0, \quad x \in \mathbb{R} \quad (4.4.1)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0. \quad (4.4.2)$$

The h function represents a source term.

We now show how to solve this more complicated situation, based on an idea known as Duhamel's principle, which involves solving the following related problem

$$v_{tt} = c^2 v_{xx}, \quad t > s, \quad x \in \mathbb{R} \quad (4.4.3)$$

$$v(x, s; s) = 0, \quad v_t(x, s; s) = h(x, s) \quad (4.4.4)$$

the solution of which, we label by $v(x, t; s)$. Now, notice that the problem (4.4.3), (4.4.4) has initial conditions prescribed at $t = s$, rather than at $t = 0$. Thus, a "shift" can reframe the problem (4.4.3), (4.4.4) into the standard form

$$w_{tt} = c^2 w_{xx}, \quad t > 0, \quad x \in \mathbb{R} \quad (4.4.5)$$

$$w(x, 0; s) = 0, \quad w_t(x, 0; s) = h(x, s) \quad (4.4.6)$$

the solution of which we label by $w(x, t; s)$ and we can write

$$v(x, t; s) = w(x, t - s; s).$$

Now, D'Alembert's solution to the homogeneous problem (4.4.5), (4.4.6) is

$$w(x, t; s) = \frac{1}{2c} \int_{x-ct}^{x+ct} h(y, s) dy$$

and hence

$$v(x, t; s) = w(x, t - s; s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} h(y, s) dy.$$

4.5 Duhamel's Principle

If $v(x, t; s)$ solves the homogeneous problem (4.4.3), (4.4.4) then the (unique) solution to the inhomogeneous problem (4.4.1), (4.4.2) is given by

Important Idea.

$$\begin{aligned} u(x, t) &= \int_0^t v(x, t; s) \, ds = \int_0^t w(x, t - s; s) \, ds \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(y, s) \, dy \, ds. \end{aligned}$$

Discussion: We verify that our u satisfies (4.4.1) and (4.4.2).

Now, it follows immediately that $u(x, 0) = 0$. Also, applying Leibniz' rule for differentiating an integral yields

$$\begin{aligned} u_t(x, t) &= v(x, t; t) + \int_0^t v_t(x, t; s) \, ds \\ &= \int_0^t v_t(x, t; s) \, ds. \end{aligned}$$

Hence, $u_t(x, 0) = 0$.

Another differentiation gives

$$\begin{aligned} u_{tt}(x, t) &= v_t(x, t; t) + \int_0^t v_{tt}(x, t; s) \, ds \\ &= h(x, t) + \int_0^t c^2 v_{xx}(x, t; s) \, ds \\ &= h(x, t) + c^2 u_{xx}(x, t) \end{aligned}$$

where we have applied (4.4.5). (We have also tacitly assumed that h is continuously differentiable). \square

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4.6 Derivation of Wave Equation

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Stretch an elastic string to length L and fix the ends. Pluck the string at time $t = 0$ and release it so that it vibrates.

Our aim is to accurately describe the motion of the string by determining the deflection $u(x, t)$ at any point x and time $t > 0$. Physical assumptions leading to our model include:

- Our string is homogenous: the mass per unit length being constant.
- Our string does not furnish any resistance to bending (ie, perfectly elastic).
- The gravitational force acting on the string can be ignored due to the large tension caused by stretching the string prior to clamping it at the end points.
- Each particle of the string moves vertically only, exhibiting small motions, so the deflection and the slope within each point of the string remains small in absolute value.



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To derive our PDE, we consider the forces acting on a small portion of our string.

- Since we have assumed our string offers no resistance to bending, the tension is tangential to the curve of the string at every point.
- Let \vec{T}_1 and \vec{T}_2 denote the tensions (resp.) at the endpoints P and Q of our small string portion.
- Due to our assumption that there is no movement in the horizontal direction, the horizontal components of the tension must be constant. Thus, we obtain

$$T_1 \cos \alpha = T_2 \cos \beta = T = \text{constant}. \quad (4.6.1)$$

- In the vertical direction there are two forces, expressly, the vertical components of \vec{T}_1 and \vec{T}_2 are (resp.) $-T_1 \sin \alpha$ and $T_2 \sin \beta$. (The minus sign occurs due to the component at P pointing downwards.)

Applying Newton's second law, we know that the resultant of the two forces is equal to the mass of the string portion ($\rho \Delta x$) times the acceleration $\partial^2 u / \partial t^2$, thus we have

$$T_2 \sin \beta - T_1 \sin \alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}.$$

Applying (4.6.1) we can thus form

$$\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \tan \alpha - \tan \beta = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}.$$

Observe that the tan values are the corresponding slopes of the string at x and $x + \Delta x$, ie

$$\tan \alpha = u_x(x, t), \quad \tan \beta = u_x(x + \Delta x, t).$$

Thus we can form

$$\frac{1}{\Delta x} [u_x(x + \Delta x, t) - u_x(x, t)] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}.$$

If we let $\Delta x \rightarrow 0$ then we obtain the PDE

$$u_{tt} = c^2 u_{xx}$$

where $c^2 := T/\rho$.

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 The solution on the whole line to

$$u_{tt} - c^2 u_{xx} = h(x, t) \tag{4.6.2}$$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \tag{4.6.3}$$

is

Important Idea.

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds \\ &+ \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(y, s) dy ds. \end{aligned} \tag{4.6.4}$$

We now establish the formula (4.6.4) directly from (4.6.2), (4.6.3). For simplicity we discuss the case $\phi \equiv 0 \equiv \psi$.

Fix a point (x_0, t_0) and consider filled triangle in the XT -plane

$$\Delta := \{(x, t) : 0 \leq t \leq t_0, x_0 - c(t_0 - t) \leq x \leq x_0 + c(t_0 - t)\}$$

and let $\partial\Delta$ denote its triangular boundary. Now, from (4.6.2) we have

$$\begin{aligned} \iint_{\Delta} f dA &= \iint_{\Delta} [u_{tt} - c^2 u_{xx}] dA \\ &= \oint_{\partial\Delta} -c^2 u_x dt - u_t dx \end{aligned}$$

by Green's theorem (with anticlockwise orientation). Let $\partial\Delta = L_0 \cup L_1 \cup L_2$ where the L_i are the three sides of Δ .

Along L_0 we have $t = 0$, $dt = 0$ so

$$\begin{aligned} \int_{L_0} -c^2 u_x dt - u_t dx &= \int_{L_0} -u_t(x, 0) dx \\ &= \int_{L_0} 0 dx \\ &= 0. \end{aligned}$$

Along L_1 we have $x = x_0 + c(t_0 - t)$ and $0 \leq t \leq t_0$. Thus $dx = -cdt$. Hence

$$\begin{aligned} \int_{L_1} -c^2 u_x dt - u_t dx &= \int_{L_1} cu_x dx + cu_t dt \\ &= c \int_{L_1} (u_x, u_t) \cdot (dx, dt) \\ &= c \int_{L_1} \nabla u \cdot d\vec{r} \\ &= c[u(x_0, t_0) - u(x_0 + ct_0, 0)] \\ &= cu(x_0, t_0) \end{aligned}$$

where we have applied the fundamental theorem of line integrals.

Similarly on L_2 we obtain

$$\int_{L_2} -c^2 u_x dt - u_t dx = cu(x_0, t_0).$$

Combining everything we obtain

$$\iint_{\Delta} f dA = 2cu(x_0, t_0)$$

and so

$$u(x_0, t_0) = \frac{1}{2c} \int_0^t \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} h(x, t) dx dt$$

which is the same as (4.6.4).



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Example.

Solve

$$u_{tt} - u_{xx} = x, \quad u(x, 0) = x^3, \quad u_t(x, 0) = \cos x.$$

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4.7 Second-Order PDEs: Classification and Solution Method

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- We will learn how to classify and solve some second-order, linear PDE.
- Methods include: factorization; and change of co-ordinates.
- PDE of a given type share common features, even if the equations look quite different.
- A more detailed treatment of the most important equations will be given later (wave, heat, Laplace etc).

Consider the PDE

$$u_{tt} = c^2 u_{xx}, \quad c = \text{constant}. \quad (4.7.1)$$


Notice that if we define the differential operator L by

$$L(u) := u_{tt} - c^2 u_{xx}$$


then (4.7.1) is just

$$L(u) = 0.$$

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We can “factor” L with the aim of obtaining two first-order PDE in the following way. Observe that

$$L(u) = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u. \quad (4.7.2)$$

This means that if we begin with a function $u(x, t)$, we compute $u_t + cu_x$. Let the result be v and then compute $v_t - cv_x$ and you should get zero. Thus, if we let

$$v = u_t + cu_x$$

then we have formed two first-order PDEs, namely

$$v_t - cv_x = 0, \quad (4.7.3)$$

$$u_t + cu_x = v. \quad (4.7.4)$$

Note that (4.7.3) is a transport-type equation, which has solution

$$v(x, t) = j(x + ct)$$

where $j(x) = v(x, 0)$ and so (4.7.4) becomes

$$u_t + cu_x = j(x + ct) \quad (4.7.5)$$

which is just an inhomogeneous transport equation. Now, we know that solution to (4.7.5) is

$$\begin{aligned} u(x, t) &= h(x - ct) + \int_0^t j(x + (s - t)c + cs) ds \\ &= h(x - ct) + \int_0^t j(x + (2s - t)c) ds \\ &= h(x - ct) + \frac{1}{2c} [G(x + (2s - t)c)]_{s=0}^{s=t} \\ &= h(x - ct) + \frac{1}{2c} [G(x + ct) - G(x - ct)] \\ &= f(x + ct) + g(x - ct) \end{aligned}$$

where $G' = j$ and we have introduced f and g to make things a little prettier.

Important Idea.

The general solution (called D'Alembert's solution) to the wave equation $u_{tt} = c^2 u_{xx}$ is

$$u(x, t) = f(x + ct) + g(x - ct).$$

4.8 Independent Learning - Reflection Method: Initial/Boundary Value Problem

Consider the problem

$$v_{tt} = c^2 v_{xx}, \quad x > 0, \quad t > 0 \quad (4.8.1)$$

$$v(x, 0) = \phi(x), \quad v_t(x, 0) = \psi(x), \quad x > 0 \quad (4.8.2)$$

$$v(0, t) = 0, \quad t \in \mathbb{R}. \quad (4.8.3)$$

This is an example of a Dirichlet problem on the half-line.

We know how to solve similar problems on the whole line, so the idea is to extend the domains of ϕ and ψ to the whole line, in such a way that the condition at $x = 0$ is automatically satisfied.

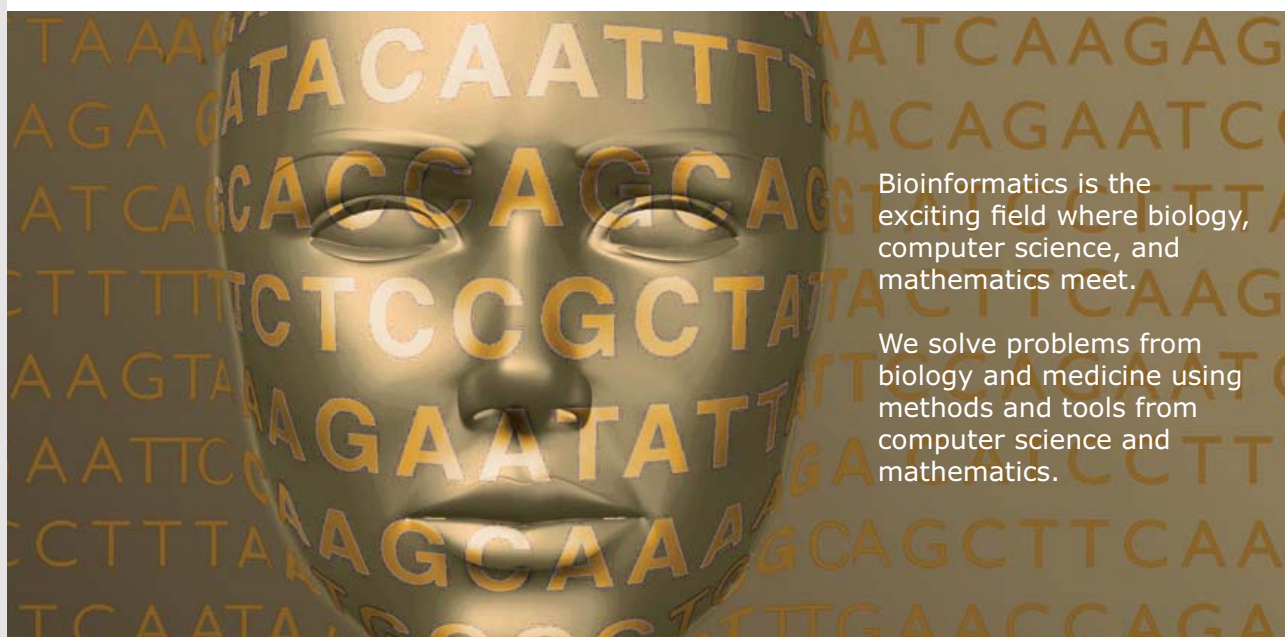
Remember that an odd extension to \mathbb{R} of a function $z = z(x)$ is defined as

$$z_o(x) := \begin{cases} z(x), & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ -z(-x), & \text{for } x < 0. \end{cases}$$



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To solve (4.8.1)–(4.8.3), consider the associated problem

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \quad (4.8.4)$$

$$u(x, 0) = \phi_o(x), \quad u_t(x, 0) = \psi_o(x), \quad x \in \mathbb{R}. \quad (4.8.5)$$

Now, the solution $u(x, t)$ can be shown to be odd in x and so $u(0, t) = 0$, so that the boundary condition (4.8.3) holds automatically.

Define $v(x, t) := u(x, t)$ for $x > 0$. Thus,

$$\begin{aligned} v(x, t) &= u(x, t) \\ &= \frac{1}{2} [\phi_o(x + ct) + \phi_o(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_o(y) \, dy. \end{aligned}$$

Let us examine this form a little more. Now, we have $x \geq 0$ and $t \geq 0$ and so $x + ct \geq 0$. Thus, $\phi_o(x + ct) = \phi(x + ct)$ above.

Now, if $x \geq 0$ and $t \geq 0$ AND $ct < x$ then: $\phi_o(x - ct) = \phi(x - ct)$; and $0 \leq x - ct \leq y \leq x + ct$, so gives $\psi_o(y) = \psi(y)$. Hence

$$u(x, t) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) \, dy \quad x > ct.$$

Alternatively, if $x \geq 0$ and $t \geq 0$ AND $x < ct$ then: $\phi_o(x - ct) = -\phi(ct - x)$; and $x - ct \leq 0 \leq x + ct$. Hence

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\phi(x + ct) - \phi(ct - x)] \\ &+ \frac{1}{2c} \left[\int_{x-ct}^0 -\psi(-y) + \int_0^{x+ct} \psi(y) \, dy \right] \quad x < ct. \end{aligned}$$

Now, a change of variable in the left integral in the previous line gives

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\phi(x + ct) - \phi(ct - x)] \\ &+ \frac{1}{2c} \left[\int_{ct-x}^0 \psi(y) \, dy + \int_0^{x+ct} \psi(y) \, dy \right] \quad x < ct. \end{aligned}$$

So, combining our cases, we have

$$v(x, t) = \begin{cases} \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) \, dy, & \text{for } x > ct, \\ \frac{1}{2} [\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(y) \, dy, & \text{for } x < ct. \end{cases}$$

4.9 Independent Learning - Reflection method: Nonhomogeneous PDE

Consider the problem

$$v_{tt} = c^2 v_{xx} + f(x, t), \quad x > 0, \quad (4.9.1)$$

$$v(x, 0) = \phi(x), \quad v_t(x, 0) = \psi(x), \quad x > 0 \quad (4.9.2)$$

$$v(0, t) = 0. \quad (4.9.3)$$

This is known as an example of a Dirichlet problem on the half-line with a nonhomogeneous PDE.

Similarly to our previous discussion on the case $f \equiv 0$. We know how to solve similar problems on the whole line, so the idea is to extend the domains of ϕ , ψ AND f in such a way that the condition at $x = 0$ is automatically satisfied.

Remember that an odd extension to \mathbb{R} of a function $z = z(x)$ is defined as

$$z_o(x) := \begin{cases} z(x), & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ -z(-x), & \text{for } x < 0. \end{cases}$$

To solve (4.9.1)–(4.9.3), consider the associated problem

$$u_{tt} = c^2 u_{xx} + f_o(x, t), \quad x \in \mathbb{R}, \quad (4.9.4)$$

$$u(x, 0) = \phi_o(x), \quad u_t(x, 0) = \psi_o(x), \quad x \in \mathbb{R}. \quad (4.9.5)$$

where f_o is the odd extension in its first variable x . Now, the solution $u(x, t)$ can be shown to be odd in x and so $u(0, t) = 0$, so the boundary condition holds automatically.

Define $v(x, t) := u(x, t)$ for $x > 0$. Thus,

$$\begin{aligned} v(x, t) &= u(x, t) \\ &= \frac{1}{2} [\phi_o(x + ct) + \phi_o(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_o(y) dy \\ &\quad + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_o(y, s) dy ds. \end{aligned}$$

Let us examine this form a little more. Now, we have $x \geq 0$ and $t \geq 0$ and so $x + ct \geq 0$. Thus, $\phi_o(x + ct) = \phi(x + ct)$ above. Furthermore, $x - c(t - s) \leq x + c(t - s)$ for $0 \leq s \leq t$.

Now, if $x \geq 0$ and $t \geq 0$ AND $ct < x$ then: $\phi_o(x - ct) = \phi(x - ct)$; and $0 \leq x - ct \leq y \leq x + ct$, so gives $\psi_o(y) = \psi(y)$. Furthermore, for all $0 \leq s \leq t$ we have $0 < x - ct + cs = x - c(t - s)$, so $f_o = f$. Hence

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy \\ &\quad + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds. \quad x > ct. \end{aligned}$$

Alternatively, if $x \geq 0$ and $t \geq 0$ AND $x < ct$ then: $\phi_o(x - ct) = -\phi(ct - x)$; and $x - ct \leq 0 \leq x + ct$ and for all $0 \leq s \leq t$ we have $c(t - s) - x < ct - x$ and so $x - c(t - s) > 0$, so $f_o = f$. Hence

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds. \\ &\quad + \frac{1}{2c} \left[\int_{x-ct}^0 -\psi(-y) dy + \int_0^{x+ct} \psi(y) dy \right] \quad x < ct. \end{aligned}$$

Now, a change of variable in the left integral in the previous line gives

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds. \\ &\quad + \frac{1}{2c} \left[\int_{ct-x}^0 \psi(y) dy + \int_0^{x+ct} \psi(y) dy \right] \quad x < ct. \end{aligned}$$

So, combining our cases, we have

$$\begin{aligned} v(x, t) &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds \\ &\quad + \begin{cases} \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy, & \text{for } x > ct, \\ \frac{1}{2} [\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(y) dy, & \text{for } x < ct. \end{cases} \end{aligned}$$

4.10 PDE with Purely Second-Order Derivatives and Classification

- We will learn how to classify and solve some second-order, linear PDE.
- Methods include: factorization; and change of co-ordinates.

- PDE of a given type share common features, even if the equations look quite different.
- A more detailed treatment of the most important equations will be given later (wave, heat, Laplace etc).

Consider the PDE

$$u_{tt} = c^2 u_{xx}, \quad c = \text{constant}. \quad (4.10.1)$$

Notice that if we define the differential operator L by

$$L(u) := u_{tt} - c^2 u_{xx}$$

then (4.10.1) is just

$$L(u) = 0.$$

We can “factor” L with the aim of obtaining two first-order PDE in the following way. Observe that

$$L(u) = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u. \quad (4.10.2)$$

This means that if we begin with a function $u(x, t)$, we compute $u_t + cu_x$. Let the result be v and then compute $v_t - cv_x$ and you should get zero.

Thus, if we let

$$v = u_t + cu_x$$

then we have formed two first-order PDEs, namely

$$v_t - cv_x = 0, \quad (4.10.3)$$

$$u_t + cu_x = v. \quad (4.10.4)$$

Note that (4.10.3) is a transport-type equation, which has solution

$$v(x, t) = j(x + ct)$$

where $j(x) = v(x, 0)$ and so (4.10.4) becomes

$$u_t + cu_x = j(x + ct) \quad (4.10.5)$$

which is just an inhomogeneous transport equation.

Now, we know that solution to (4.10.5) is

$$\begin{aligned} u(x, t) &= h(x - ct) + \int_0^t j(x + (s - t)c + cs) \, ds \\ &= h(x - ct) + \int_0^t j(x + (2s - t)c) \, ds \\ &= h(x - ct) + \frac{1}{2c} [G(x + (2s - t)c)]_{s=0}^{s=t} \\ &= h(x - ct) + \frac{1}{2c} [G(x + ct) - G(x - ct)] \\ &= f(x + ct) + g(x - ct) \end{aligned}$$

where $G' = j$ and we have introduced f and g to make things a little prettier.

Important Idea.

The general solution to the wave equation $u_{tt} = c^2 u_{xx}$ is

$$u(x, t) = f(x + ct) + g(x - ct).$$

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Motivated by the form of solution to the wave equation, we consider the more general PDE

$$au_{xx} + bu_{xy} + cu_{yy} = 0, \quad a, b, c = \text{constants.} \quad (4.10.6)$$

Assume a solution to (4.10.6) is of the form

$$u(x, y) = f(mx + y)$$

for some constant m to be determined.

Differentiating our assumed form and substituting into our PDE (4.10.6) we have

$$0 = am^2 f'' + bm f'' + c f'' = 0.$$

Thus we have a quadratic equation that determines m with

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- If $b^2 - 4ac > 0$ then we obtain two real and unequal roots (say, m_1 and m_2) and we term our PDE (4.10.6) as HYPERBOLIC, with the general solution of the form

$$u(x, y) = f(m_1 x + y) + g(m_2 x + y).$$

- If $b^2 - 4ac < 0$ then we obtain (two) complex roots (say, m_1 and m_2) and we term our PDE (4.10.6) as ELLIPTIC, with the general solution of the form

$$u(x, y) = f(m_1 x + y) + g(m_2 x + y).$$

- If $b^2 - 4ac = 0$ then we obtain two real and equal roots (say, $m = m_1 = m_2$) and we term our PDE (4.10.6) as PARABOLIC, with the general solution of the form

$$u(x, y) = f(mx + y) + xg(mx + y).$$

If we assume a solution to (4.10.6) is of the form $u(x, y) = f(x + my)$ for some constant m to be determined then we can rerun the previous argument to obtain the quadratic equation $cm^2 + bm + a = 0$ so that

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}.$$

- If $b^2 - 4ac > 0$ then we obtain two real and unequal roots (say, m_1 and m_2) and we term our PDE (4.10.6) as HYPERBOLIC, with the general solution of the form

$$u(x, y) = f(x + m_1 y) + g(x + m_2 y).$$

- If $b^2 - 4ac < 0$ then we obtain (two) complex roots (say, m_1 and m_2) and we term our PDE (4.10.6) as ELLIPTIC, with the general solution of the form

$$u(x, y) = f(x + m_1 y) + g(x + m_2 y).$$

- If $b^2 - 4ac = 0$ then we obtain two real and equal roots (say, $m = m_1 = m_2$) and we term our PDE (4.10.6) as PARABOLIC, with the general solution of the form $u(x, y) = f(x + my) + yg(x + my)$.

Example.

Classify and solve:

$$u_{xx} + u_{xy} - 2u_{yy} = 0.$$

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Example.

Classify and solve Laplace's equation:

$$u_{xx} + u_{yy} = 0.$$

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Example.

Classify and solve:

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0.$$

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4.11 Classifying Second-Order PDE with First-Order Derivatives

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For the second order PDE

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + hu = 0 : \quad (4.11.1)$$

- If $b^2 - 4ac > 0$ then we term our PDE (4.11.1) as HYPERBOLIC.
- If $b^2 - 4ac < 0$ then we term our PDE (4.11.1) as ELLIPTIC.
- If $b^2 - 4ac = 0$ then we term our PDE (4.11.1) as PARABOLIC.

Example.

Classify the PDEs

$$\begin{aligned}u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u &= 0. \\yu_{xx} - 4u_{xy} + 4xu_{yy} &= 0.\end{aligned}$$

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Chapter 5

The Heat Equation

5.1 Introduction

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- We will discover a number of approaches to solving a famous PDE known as the heat (diffusion) equation

$$u_t = ku_{xx}.$$

- The discussion involves problems on the whole line and the half line.
- The methods are in stark contrast to the techniques applied in solving the wave equation $u_{tt} = c^2u_{xx}$.
- The ideas will guide us on solving a range of other second-order problems.

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5.2 Diffusion on the Whole Line

Let's consider the following diffusion problem on the whole line

$$u_t = ku_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \quad (5.2.1)$$

$$u(x, 0) = \phi(x) \quad (5.2.2)$$

where k is a constant and ϕ is a given function.

We shall form an explicit formula for the solution by solving a modified problem.

The solution will be shown to be

$$u(x, t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy.$$

We'll need five basic properties, known as *invariance properties* of the diffusion equation (5.2.1), namely

1. If $u(x, t)$ is a solution to (5.2.1) then for each fixed y , the translate $u(x - y, t)$ is also a solution;
2. If u is a solution then every derivative of u is also a solution (ie, u_x etc is a solution);
3. Every linear combination of solutions is also a solution (linearity);
4. If $S(x, t)$ is a solution then so is its integral. Thus, $S(x - y, t)$ is a solution and so is

$$\int_{-\infty}^{\infty} S(x - y, t)g(y) dy$$

for each function g that makes the integral converge.

5. If $u(x, t)$ is a solution then for each $a > 0$ the dilated function $u(\sqrt{a}x, at)$ is also a solution.

5.3 The Modified Problem

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We shall solve the following modified problem

$$Q_t = kQ_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \quad (5.3.1)$$

$$Q(x, 0) = 1, \quad x > 0; \quad Q(x, 0) = 0, \quad x < 0. \quad (5.3.2)$$

We will then construct all solutions to the original problem (5.2.1), (5.2.2) via the translation and integral properties of solutions to (5.3.3).

Let us construct Q via the particular form

$$Q(x, t) = g\left(\frac{x}{\sqrt{4kt}}\right).$$

WHY?? Well, the dilation property of (5.3.1) means that the dilations $x \rightarrow \sqrt{a}x$, $t \rightarrow at$ don't affect the PDE (5.3.1). In addition, these do not affect (5.3.2).

How can this happen? $Q(x, t)$ must be a function of the ratio x/\sqrt{t} .

If we let

$$X := \sqrt{a}x, \quad \text{and } T := at$$

then notice that the ratio

$$\frac{X^2}{T} = \frac{ax^2}{at} = \frac{x^2}{t}$$

or

$$\frac{X}{\sqrt{T}} = \frac{\sqrt{a}x}{\sqrt{at}} = \frac{x}{\sqrt{t}}.$$

This kinds of transformations are known as *invariant* with respect to the PDE (5.3.3).

Let us solve (5.3.1), (5.3.2) from the form

$$Q(x, t) = g\left(\frac{x}{\sqrt{4kt}}\right)$$

by determining g . (The $\sqrt{4k}$ is there for a little convenience later.)

The idea is to form a ODE to find g . Now, computing derivatives via the chain rule we have

$$\begin{aligned} Q_t &= -\frac{1}{2t} \frac{x}{\sqrt{4kt}} g'\left(\frac{x}{\sqrt{4kt}}\right) \\ Q_x &= \frac{1}{\sqrt{4kt}} g'\left(\frac{x}{\sqrt{4kt}}\right) \\ Q_{xx} &= \frac{1}{4kt} g''\left(\frac{x}{\sqrt{4kt}}\right). \end{aligned}$$

Hence

$$0 = Q_t - kQ_{xx} = \frac{1}{t} \left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p) \right]$$

and so we have formed the ODE

$$g'' + 2pg' = 0. \tag{5.3.3}$$

To solve (5.3.3), we can let $z = g'$ so that we have the first order ODE

$$z' = -2pz$$

which has exponential solutions

$$z(p) = Ae^{-p^2}, \quad A = \text{constant}.$$

Hence, integration from $s = 0$ to $s = p$ yields

$$g(p) = A \int_0^p e^{-s^2} ds + B, \quad B := g(0) = \text{constant}.$$

Thus, a general representation for Q is

$$Q(x, t) = g(p) = A \int_0^{x/\sqrt{4kt}} e^{-s^2} ds + B.$$

Now, this solution only make sense for $t > 0$.

Let us determine the constants A and B by applying the conditions (5.3.2) via

$$\lim_{t \rightarrow 0^+} Q(x, t) = \begin{cases} 1 = A \int_0^\infty e^{-s^2} ds + B, & \text{for } x > 0, \\ 0 = A \int_0^{-\infty} e^{-s^2} ds + B, & \text{for } x < 0. \end{cases}$$

so we have two simultaneous equations

$$\begin{aligned} A\frac{\sqrt{\pi}}{2} + B &= 1 \\ -A\frac{\sqrt{\pi}}{2} + B &= 0. \end{aligned}$$

Hence $A = 1/\sqrt{\pi}$ and $B = 1/2$. Thus

$$Q(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2} ds, \quad t > 0.$$

Now, we know that derivative of $\partial Q/\partial x$ is also a solution to (5.2.1). Define

$$S := \partial Q/\partial x.$$

Now, given any suitable ϕ , consider

$$u(x, t) = \int_{-\infty}^{\infty} S(x-y, t)\phi(y) dy, \quad t > 0$$

which, by the translate and integral property, is also a solution to (5.2.1). It remains to show that (5.2.2) holds. Consider

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{\infty} Q_x(x-y, t)\phi(y) dy \\ &= - \int_{-\infty}^{\infty} \frac{\partial}{\partial y} [Q(x-y, t)]\phi(y) dy \\ &= \int_{-\infty}^{\infty} Q(x-y, t)\phi'(y) dy - Q(x-y, t)\phi(y) \Big|_{y \rightarrow -\infty}^{y \rightarrow \infty} \end{aligned}$$

where we have integrated by parts.

Now, assuming the previous limits vanish and, in particular, that $\phi(y)$ is zero for large $\pm y$, then

$$\begin{aligned} u(x, 0) &= \int_{-\infty}^{\infty} Q(x-y, 0)\phi'(y) dy \\ &= \int_{-\infty}^x \phi'(y) dy \\ &= \phi(x). \end{aligned}$$

Above we have used the initial data for Q .

Thus we conclude that our u really is the solution to (5.2.1), (5.2.2), where

$$S := \frac{\partial Q}{\partial x} = \frac{1}{\sqrt{4k\pi t}} e^{-x^2/4kt}$$

and so

$$u(x, t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy.$$

View this lesson on YouTube [26]

Example.

Solve

$$u_t = u_{xx}, \quad \phi(x) = \begin{cases} 1, & \text{for } |x| < 2, \\ 0, & \text{for } |x| > 2, \end{cases}$$

in terms of error functions.

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5.4 Independent Learning - Heat Equation: Inhomogenous Case

Consider the inhomogeneous problem associated with the heat equation

$$u_t - ku_{xx} = h(x, t), \quad t > 0, \quad x \in \mathbb{R} \quad (5.4.1)$$

$$u(x, 0) = 0. \quad (5.4.2)$$

The h function represents a source term.

We now show how to solve this more complicated situation, based on an idea known as Duhamel's principle, which involves solving the following related problem

$$v_t = kv_{xx}, \quad t > s, \quad x \in \mathbb{R} \quad (5.4.3)$$

$$v(x, s; s) = h(x, s) \quad (5.4.4)$$

the solution of which, we label by $v(x, t; s)$.

Now, notice that the problem (5.4.3), (5.4.4) has initial conditions prescribed at $t = s$, rather than at $t = 0$. Thus, a "shift" can reframe the problem (5.4.3), (5.4.4) into the standard form

$$w_t = kw_{xx}, \quad t > 0, \quad x \in \mathbb{R} \quad (5.4.5)$$

$$v(x, 0; s) = h(x, s) \quad (5.4.6)$$

the solution of which we label by $w(x, t; s)$ and we can write

$$v(x, t; s) = w(x, t - s; s).$$

Now, D'Alembert's solution to the homogeneous problem (5.4.5), (5.4.6) is

$$w(x, t; s) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} h(y, s) dy$$

and hence

$$v(x, t; s) = w(x, t - s; s) = \frac{1}{\sqrt{4k\pi(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-s)}} h(y, s) dy.$$

5.5 Independent Learning - Duhamel's Principle

If $v(x, t; s)$ solves the homogeneous problem (5.4.3), (5.4.4) then the (unique) solution to the inhomogeneous problem (5.4.1), (5.4.2) is given by

Important Idea.

$$\begin{aligned} u(x, t) &= \int_0^t v(x, t; s) ds = \int_0^t w(x, t-s; s) ds \\ &= \int_0^t \frac{1}{\sqrt{4k\pi(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-s)}} h(y, s) dy ds. \end{aligned}$$



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5.6 Solving Heat Equation on Half Line

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Example.

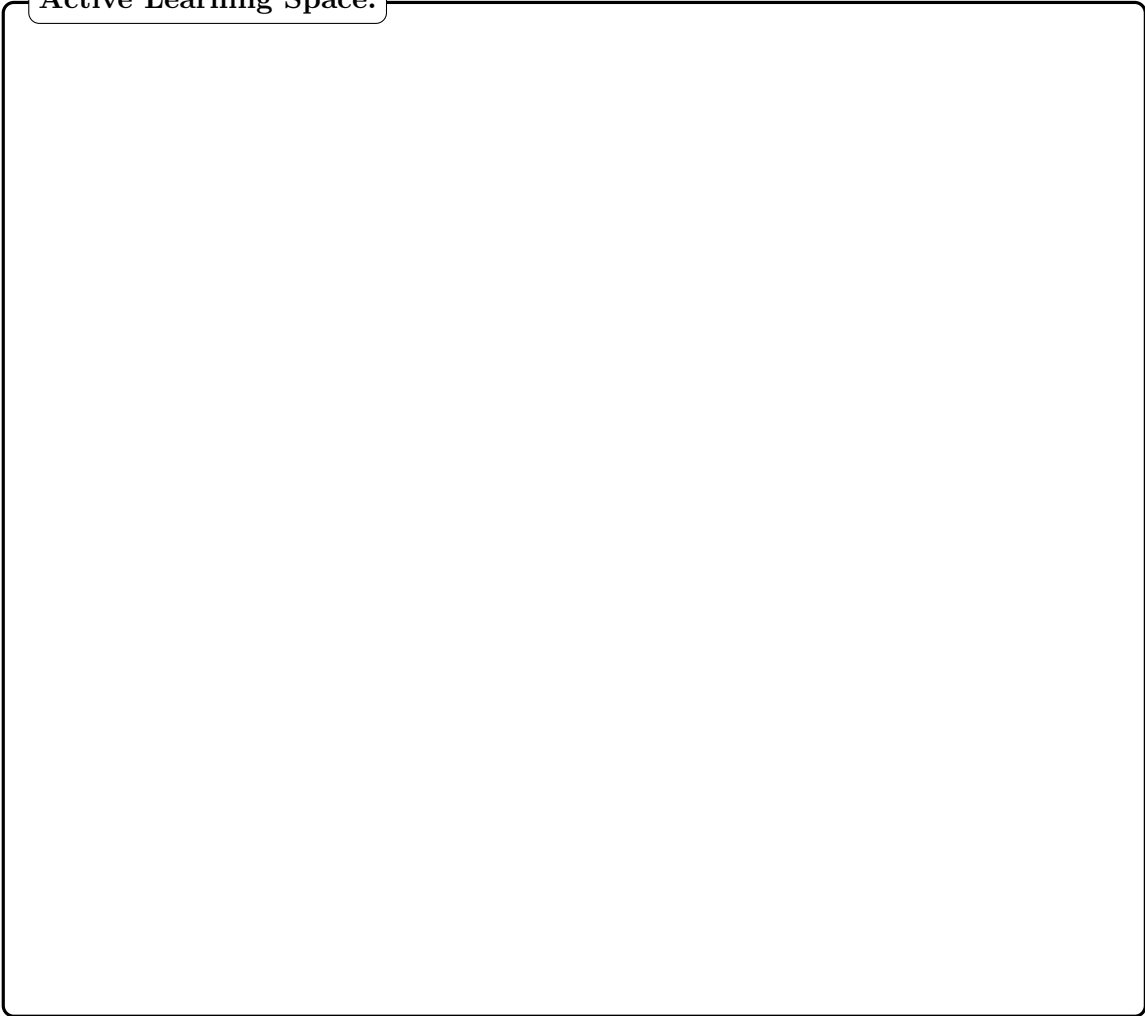
Consider the diffusion problem on the half line $x > 0$:

$$v_t = v_{xx}, v(x, 0) = \phi(x), v(0, t) = 0$$

- Extend ϕ to \mathbb{R} as an odd function ϕ_o
- Solve the problem on the whole line $u_t = u_{xx}, u(x, 0) = \phi_o(x)$
- Explain how to obtain the solution to our original problem.

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5.7 Derivation of Heat Equation in 1-Dimension

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In the early 1800s, J. Fourier began a mathematical study of heat. A deeper understanding of heat flow had significant applications in science and within industry. A basic version of Fourier's efforts is the problem

$$\alpha^2 u_{xx} = u_t; \quad u(0, t) = 0 = u(L, t); \quad u(x, 0) = f(x)$$

where: $u(x, t)$ is the temperature at position x at time t ; α^2 is a constant; and f is a given function. Fourier's analysis resulted in the following solution form

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-t(\alpha n \pi / L)^2} \sin \frac{n \pi x}{L}$$

provided f could be written in the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{L}.$$

This prompted Fourier to form a method for expressing functions as infinite sums of sines (and/or cosines), called Fourier series.

Suppose we have a thin bar of length L wrapped around the x -axis, so that $x = 0$ and $x = L$ are the ends of the bar. We assume that the bar is of homogeneous material, is straight and has uniform cross-sections. Further, suppose that the sides of the bar are perfectly insulated so that no heat passes through them.

Since our bar is thin, the temperature u can be considered as constant on any given cross-section and so depends on the horizontal position along the x -axis. Hence u is a function only of position x and time t , ie

$$u = u(x, t)$$

We now show how to form the heat equation

$$u_t = \alpha^2 u_{xx} \tag{5.7.1}$$

where $\alpha^2 := \kappa / \rho c$ is called the thermal diffusivity (in (length)²/time) and κ , ρ and c are positive constants that depend on the material of the bar.

Consider a section D of the bar, with ends at x_0 and x_1 . Now the total amount of heat $H = H(t)$ in D (say, in calories) is

$$H(t) = \int_{x_0}^{x_1} c \rho u(x, t) dx.$$

Differentiating (using Leibniz rule) we obtain

$$\frac{dH}{dt} = c \rho \int_{x_0}^{x_1} u_t(x, t) dx.$$

Above, c is the specific heat of the material (the quantity of (heat) energy needed to raise the temperature of a unit quantity of material by 1 degree of temperature) and ρ is the density of the material.

Now since the sides of the bar are insulated, the only way heat can flow into or out of D is through the ends at x_0 and x_1 . Fourier's law of heat flow says that heat flows from hotter regions to colder regions and the flow rate is proportional to u_x .

Now, the net rate of change of heat H in D is just the rate at which heat enters D minus the rate at which heat leaves D , ie

$$\frac{dH}{dt} = -\kappa u_x(x_0, t) - (-\kappa u_x(x_1, t)).$$

The minus sign appears in the above two terms since there will be a positive flow of heat from the left to right only if the temperature is greater to the left of $x = x_0$ than to the right (in this case, $u_x(x_0, t)$ will be negative). Above, κ is known as the thermal conductivity.

Now, simplifying the above and applying the fundamental theorem of calculus we obtain

$$\begin{aligned} \frac{dH}{dt} &= \kappa u_x(x_1, t) - \kappa u_x(x_0, t) \\ &= \kappa \int_{x_0}^{x_1} u_{xx}(x, t) dx. \end{aligned}$$

Comparing our two expressions for dH/dt we form the relationship

$$c\rho \int_{x_0}^{x_1} u_t(x, t) dx = \kappa \int_{x_0}^{x_1} u_{xx}(x, t) dx$$

and differentiating both sides with respect to x_1 we obtain

$$c\rho u_t = \kappa u_{xx}. \tag{5.7.2}$$

Since the above arguments work for all intervals from x_0 to x_1 and for all $t > 0$ it follows that the above PDE is satisfied for our interval of interests: from $x = 0$ to $x = L$ (and all $t > 0$).

The PDE (5.7.2) essentially describes a fundamental physical balance: the rate at which heat flows into any portion of the bar is equal to the rate at which heat is absorbed into that portion of the bar.

Hence the two terms in (5.7.2) are sometimes referred to as: the "absorption term" ($c\rho u_t$) and the "flux term" (κu_{xx}).

View this lesson on YouTube [29]

Example.

Consider

$$u_{xx} + 2u_{yy} = 0.$$

Assuming $u = g(x/y)$, formulate the ODE that g satisfies.

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5.8 Similarity Solutions to PDE

View this lesson on YouTube [30]

Where are we going?

- We will discover a new method of solving PDE, known as the “similarity solution” approach.
- In contrast to the method of “separation of variables”, the new method actually involves connecting or combining the variables in a special way.
- Such techniques are very powerful, enabling us to solve a range of linear and nonlinear problems.
- The basic idea is to determine a set of “invariant stretching transformations” that leaves the PDE in question unchanged. In this way we can reduce the problem to an ODE which may be solvable.

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We have already seen the idea of “invariant stretching transformations” at work when we solved the heat/diffusion equation

$$u_t = ku_{xx} \tag{5.8.1}$$

where $k > 0$ is a constant. If we let

$$X := \sqrt{ax}, \quad \text{and } T := at$$

then $u(X, T)$ solves (5.8.1) and notice that the ratio

$$\frac{X^2}{T} = \frac{ax^2}{at} = \frac{x^2}{t}$$

or

$$\frac{X}{\sqrt{T}} = \frac{\sqrt{ax}}{\sqrt{at}} = \frac{x}{\sqrt{t}}.$$

This kinds of transformations are known as *invariant* with respect to the PDE. (5.8.1).

Based on the above, we assumed solutions to (5.8.1) were of the form like

$$u(x, t) = g\left(\frac{x}{\sqrt{4kt}}\right)$$

and then (5.8.1) reduced to an ODE

$$g'' + 2pg' = 0, \quad p = x/\sqrt{4kt} \tag{5.8.2}$$

which we could solve to obtain

$$u(x, t) = g(p) = A \int_0^{x/\sqrt{4kt}} e^{-s^2} ds + B.$$

We now show how to go further!

Consider, (say) the PDE

$$L(u) = 0.$$

Consider the transformations

$$X = a^\alpha x, \quad Y = a^\beta y, \quad \bar{u} = a^\gamma u.$$

From the PDE under consideration we wish to determine values for α, β, γ such that

$$L(\bar{u}) = 0.$$

Then construct solutions of form

$$u(x, y) = y^{-r} g(xy^{-s})$$

for some r and s .

Form and solve an appropriate ordinary differential equation.

The procedure is best explained via examples.

Example.

Solve, by applying the similarity solution method:

$$\begin{aligned} u_t &= ku_{xx}; \\ u(x, 0) &= 0, \quad x > 0; \\ u(x, t) &\rightarrow 0, \quad x \rightarrow \infty; \quad u_x(0, t) = N, \quad t > 0. \end{aligned}$$

Solution: #1. Determine a set of stretching (dilation) transformations under which the PDE is invariant.

Consider

$$\bar{X} = a^\alpha x; \quad \bar{T} = a^\beta t; \quad \bar{u}(\bar{X}, \bar{T}) = a^\gamma u(x, t).$$

Now, by the chain rule we have

$$\bar{u}_{\bar{T}} = a^\gamma u_t t_{\bar{T}} = a^{\gamma-\beta} u_t.$$

Similarly, $\bar{u}_{\bar{X}} = a^\gamma u_x x_{\bar{X}} = a^{\gamma-\alpha} u_x$ and $\bar{u}_{\bar{X}\bar{X}} = a^{\gamma-2\alpha} (u_x)_{\bar{X}} = a^{\gamma-2\alpha} u_{xx}$. Hence

$$\begin{aligned} \bar{u}_{\bar{T}} - k\bar{u}_{\bar{X}\bar{X}} &= a^{\gamma-\beta} u_t - ka^{\gamma-2\alpha} u_{xx} \\ &= 0 \end{aligned}$$

when $\gamma - \beta = \gamma - 2\alpha$. That is, our PDE is invariant under the stretching transformations when

$$\beta = 2\alpha$$

and γ is arbitrary. This means that when $u = u(x, t)$ solves the PDE then so does $a^\gamma u(a^\alpha x, a^\beta t)$.

Example.

#2. Determine s and t such that $\bar{X}\bar{T}^s = xt^s$ and $\bar{u}\bar{T}^r = ut^r$. We have

$$\begin{aligned}\bar{X}\bar{T}^s &= a^\alpha x (a^\beta t)^s = a^\alpha x (a^{2\alpha} t)^s \\ &= a^{\alpha(1+2s)} xt^s \\ &= xt^s\end{aligned}$$

for $s = -1/2$. Hence $xt^{-1/2}$ forms an absolute invariant and we let $p = xt^{-1/2}$. Now

$$\begin{aligned}\bar{u}\bar{T}^r &= a^\gamma \bar{u} (a^\beta t)^r = a^\gamma u (a^{2\alpha} t)^r \\ &= a^{\gamma+2\alpha r} ut^r \\ &= ut^r\end{aligned}$$

for $r = -\gamma/2\alpha$.

#3. Form $u(x, t) = t^{-r} g(xt^s) = t^{-r} g(p)$.

For our situation, we have

$$u(x, t) = t^{\gamma/2\alpha} g(xt^{-1/2}) = t^{\gamma/2\alpha} g(p).$$

We can determine γ and α from our boundary conditions. Observe

$$\begin{aligned}u_x(0, t) &= t^{\gamma/2\alpha-1/2} g'(xt^{-1/2})|_{x=0} \\ &= t^{\gamma/2\alpha-1/2} g'(0) \\ &= N\end{aligned}$$

if $\gamma/2\alpha = 1/2$. Thus, $u(x, t) = t^{1/2} g(xt^{-1/2}) = t^{1/2} g(p)$. Furthermore, we have formed the boundary condition $g'(0) = N$. Now,

$$\begin{aligned}u(x, t) &= t^{1/2} g(xt^{-1/2}) \\ &\rightarrow 0\end{aligned}$$

as $x \rightarrow \infty$ leads us to $g(p) \rightarrow 0$ as $p \rightarrow \infty$.

All we need now is an ODE in g . We compute the derivatives that appear in our original PDE, namely

$$\begin{aligned}u_t &= \left[t^{1/2} g(xt^{-1/2}) \right]_t \\ &= \frac{1}{2} t^{-1/2} g + t^{1/2} \left(-\frac{x}{2t^{3/2}} \right) g' \\ &= \frac{t^{-1/2}}{2} [g - pg'].\end{aligned}$$

Also,

$$u_{xx} = \left[t^{1/2} g(xt^{-1/2}) \right]_{xx} = t^{-1/2} g''.$$

Example.

Thus, our PDE becomes

$$\begin{aligned} 0 &= u_t - k u_{xx} \\ &= \frac{t^{-1/2}}{2} [g - pg'] - kt^{-1/2} g'' \end{aligned}$$

that, is

$$2kg'' + pg' - g = 0. \quad (5.8.3)$$

Now, we notice that one set of solutions to this ODE is $g_1(p) = C_1 p$, where C_1 is a constant.

We may construct a second solution via the reduction of order method, via

$$g_2(p) = ph(p)$$

where h is a function to be determined. Calculate g'_2, g''_2 etc and substitute into (5.8.3). Now, our ODE (5.8.3) becomes

$$2kph'' + (4k + p^2)h' = 0$$

so that

$$\begin{aligned} h'(p) &= C_2 e^{-\int (4k+p^2)/2kp \, dp} \\ &= C_2 e^{-\int (2/p+p/2k) \, dp} \\ &= C_2 e^{-(2 \ln p + p^2/4k)} \\ &= \frac{C_2}{p^2} e^{-p^2/4k}. \end{aligned}$$

Hence

$$h(\infty) - h(p) = \int_p^\infty \frac{C_2}{s^2} e^{-s^2/4k} \, ds$$

and denote $M := h(\infty)$. Now, integrating by parts we have

$$h(p) = M - C_2 \left(\frac{1}{p} e^{-p^2/4k} - \frac{1}{2k} \int_p^\infty e^{-s^2/4k} \, ds \right).$$

Thus

$$g(p) = (M + C_1)p - C_2 \left(e^{-p^2/4k} - \frac{p}{2k} \int_p^\infty e^{-s^2/4k} \, ds \right).$$

Applying our boundary conditions we see that $g(\infty) = 0$ for $M + C_1 = 0$, hence

$$g(p) = -C_2 \left(e^{-p^2/4k} - \frac{p}{2k} \int_p^\infty e^{-s^2/4k} \, ds \right).$$

Example.

In addition

$$N = g'(0) = \frac{C_2}{2k} \int_0^\infty e^{-s^2/4k} ds.$$

If we let $z = s/2\sqrt{k}$ then $dz = ds/2\sqrt{k}$ and so

$$\begin{aligned} N &= \frac{C_2}{\sqrt{k}} \int_0^\infty e^{-z^2} dz \\ &= \frac{C_2}{\sqrt{k}} \frac{\sqrt{\pi}}{2}. \end{aligned}$$

Hence

$$C_2 = \frac{2N\sqrt{k}}{\sqrt{\pi}}.$$

Thus

$$g(p) = -\frac{2N\sqrt{k}}{\sqrt{\pi}} \left[e^{-p^2/4k} - \frac{p}{2k} \int_p^\infty e^{-s^2/4k} ds \right]$$

#4. We have solved our ODE!! Finally, we have

$$\begin{aligned} u(x, t) &= t^{1/2} g(xt^{-1/2}) \\ &= -t^{1/2} \frac{2N\sqrt{k}}{\sqrt{\pi}} \left[e^{-x^2/4kt} - \frac{x}{2k\sqrt{t}} \int_{x/\sqrt{t}}^\infty e^{-s^2/4kt} ds \right]. \end{aligned}$$

If desired, this can be written in terms of a complementary error function.

Similarity solution method summary

1. Determine a set of stretching transformations

$$\bar{X} = a^\alpha x; \quad \bar{T} = a^\beta t; \quad \bar{u} = a^\gamma u$$

under which the PDE is invariant.

2. Determine s and r such that

$$\bar{X}\bar{T}^s = xt^s, \quad \bar{u}\bar{T}^r = ut^r.$$

3. Form

$$u(x, t) = t^{-r} g(xt^s) = t^{-r} g(p)$$

and an appropriate ODE / boundary value problem for g in p .

4. Solve (if possible).

Chapter 6

Laplace Transforms

6.1 Introduction

View this lesson on YouTube [31]

Many practical problems involve mechanical or electrical systems acted on by discontinuous he ODEs used to model these processes can be a little awkward to solve by using our previous methods.

We shall learn a new method based on the concept of **integral transforms**.

Let $f(t)$ be defined for $t \geq 0$. We define and denote the **Laplace transform** of f by $f = f(t)$ by

$$\mathcal{L}\{f(t)\} = F(s) := \int_0^{\infty} e^{-st} f(t) dt. \quad (6.1.1)$$

Provided this improper integral converges.

If f is a **piecewise continuous** function and $|f(t)| \leq Ke^{At}$ for all $t \geq M$ for some constants $A, K > 0, M > 0$ (i.e. for all t sufficiently large), then (6.1.1) converges for all $s > A$

A function f is **piecewise continuous** on an interval I if we can slice up the interval into a finite number of subintervals $I_i = [t_i, t_i + 1]$ such that:

f is continuous on each open subinterval $(T_i, t_i + 1)$; f approaches a finite limit as the ends of each subinterval are approached from within the subinterval.

It is not essential that f be defined at the endpoints of our subintervals.

Example.

If $f(t) := e^{-at}$ for all $t \geq 0$ where a is constant then compute

$$\mathcal{L}\{f(t)\} = F(s)$$

for $s > -a$.

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Based on the above, we observe $\mathcal{L}\{e^{ibt}\} =$

Example.

If $f(t) := \cos(bt)$ for all $t \geq 0$ where b is constant then compute

$$\mathcal{L}\{f(t)\} = F(s).$$

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Example.

The Laplace transform of a function $f(t)$ is

$$\mathcal{L}\{f(t)\} = F(s) := \int_0^{\infty} e^{-st} f(t) dt.$$

Apply Leibniz rule by differentiating under the integral sign to obtain

$$\mathcal{L}\{tf(t)\} = -F'(s).$$

Hence compute the Laplace transform of te^{-at} .

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Example.

(Challenge) Show (perhaps via induction)

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

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6.2 Inverse Laplace Transforms

To complement our Laplace transform, we define the inverse Laplace transform of $F(s)$ as \mathcal{L}^{-1} via

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

so that $F(s)$ is the Laplace transform of $f(t)$.

Both \mathcal{L} and \mathcal{L}^{-1} are linear operators in the sense that

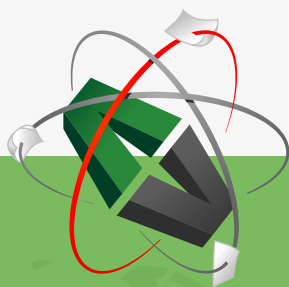
$$\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha\mathcal{L}\{f_1(t)\} + \beta\mathcal{L}\{f_2(t)\}$$

$$\mathcal{L}^{-1}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha\mathcal{L}^{-1}\{f_1(t)\} + \beta\mathcal{L}^{-1}\{f_2(t)\}$$

The above properties will be extremely useful in our investigation.

It is possible to write down a general form of the inverse Laplace transform, but it requires knowledge of the theory of functions of complex variables. Our above definition will prove useful for our discussion.

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6.3 Using Tables to Calculate Transforms

For practical purposes, it is common to use a table of transforms to calculate (inverse) Laplace transforms. Although the Laplace transform can be sometimes quickly calculated by integration, a real need for a table arises when calculating inverse transforms, as we do not have an explicit form for the inverse transform (and would have to resort to a "guess and check" method by calculating Laplace transforms of certain functions).

In the following table, a and b are any real numbers, c is any non-negative real number, m is any non-negative integer, ν is any real number larger than -1 , $\Gamma(\nu + 1) = \int_0^\infty e^{-x} x^\nu dx$ and

$$f(t) := \begin{cases} 0, & \text{for } t < 0, \\ 1/2, & \text{for } t = 0 \\ 1, & \text{for } t > 0. \end{cases}$$

Table of Laplace Transforms

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^m	$\frac{m!}{s^{m+1}}$
$t^\nu (\nu > 1)$	$\frac{\Gamma(\nu+1)}{s^{\nu+1}}$
e^{-st}	$\frac{1}{(s+a)}$
$\sin(bt)$	$\frac{b}{(s^2+b^2)}$
$\cos(bt)$	$\frac{s}{(s^2+b^2)}$
$\sinh(bt)$	$\frac{b}{(s^2-b^2)}$
$\cosh(bt)$	$\frac{s}{(s^2-b^2)}$
$\sin(bt)-bt \cos(bt)$	$\frac{2b^3}{(s^2+b^2)^2}$
$\sin(bt)+bt \cos(bt)$	$\frac{2bs^2}{(s^2+b^2)^2}$
$t \sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$u(t-c)$	$\frac{e^{-cs}}{s}$
$e^{-at}g(t)$	$G(s+a)$
$tg(t)$	$-G'(s)$
$u(t-c)g(t-c)$	$e^{-cs}G(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2G(0) - sg'(0) - g''(0)$
$\int_0^t g(\tau) d\tau$	$\frac{G(s)}{s}$

Example.

If

$$F(s) := \frac{1}{s+4} + \frac{s}{s^2-9}$$

then use a table to calculate

$$\mathcal{L}^{-1}\{F(s)\}.$$

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6.4 First Shifting Theorem

View this lesson on YouTube [32]

The first shifting theorem provides a convenient way of calculating the Laplace transforms of functions that are of the form $f(t) := e^{-at}g(t)$ where a is a constant and g is a given function. If $G(s) = \mathcal{L}\{g(t)\}$ then

$$\mathcal{L}\{e^{-at}g(t)\} = G(s + a)$$

$$\mathcal{L}^{-1}\{G(s + a)\} = e^{-at}g(t).$$

Usually a little bit of algebraic manipulation is required to use this theorem. Observe that we are just “replacing” s with $(s + a)$ in the transform.

Example.

What is

$$\mathcal{L}\{e^{4t}t^6\}?$$

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Example.

Calculate the Laplace transform of $(t + 1)^2e^t$.

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Example.

Calculate the inverse Laplace transform of $\frac{1}{(s+1)^2}$.

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Example.

Calculate the inverse Laplace transform of $s - 2s^2 - 4s + 5$.

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6.5 Introduction to Heaviside Functions

View this lesson on YouTube [33]

We define the unit step function $u(t)$ as

$$u(t) := \begin{cases} 0, & \text{for } t < 0, \\ 1/2, & \text{for } t = 0 \\ 1, & \text{for } t > 0. \end{cases}$$

and in general, define $u(t - a)$ (for $a \geq 0$), as:

$$u(t - a) := \begin{cases} 0, & \text{for } t < 0, \\ 1/2, & \text{for } t = 0 \\ 1, & \text{for } t > 0. \end{cases}$$

The unit step function is a typical function used in engineering to model phenomena that are “switched” on and off.

Do not be too concerned with the value of the step function at a . Different textbooks use various definitions at a , but it is actually not a big deal. We will use convention that at $t = a$, $u(t - a) = 1/2$.

If we multiply $u(t - a)$ with a function $f(t)$ then we can model all sorts of effects.

Example.

Sketch the graphs of the functions

$$g(t) := 1 + u(t - 1), \quad q(t) := 1 - 2u(t - 2) \text{ for } t \geq 0.$$

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Example.

Sketch the graphs of the functions

$$z(t) := u(t-1) + u(t-2), \quad w(t) := u(t-1) - 2u(t-2).$$

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Example.

Sketch the functions

$$z(t) := 1 - u(t-2)(t^2), \quad w(t) := 1 - u(t-2)(t-1)^2.$$

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Example.

If

$$f(t) := \begin{cases} t, & \text{for } 0 \leq t < 4, \\ 0, & \text{otherwise.} \end{cases}$$

then write f in terms of $[t - 4]u(t - 4)$.

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6.6 Second Shifting Theorem: Laplace Transforms

View this lesson on YouTube [34]

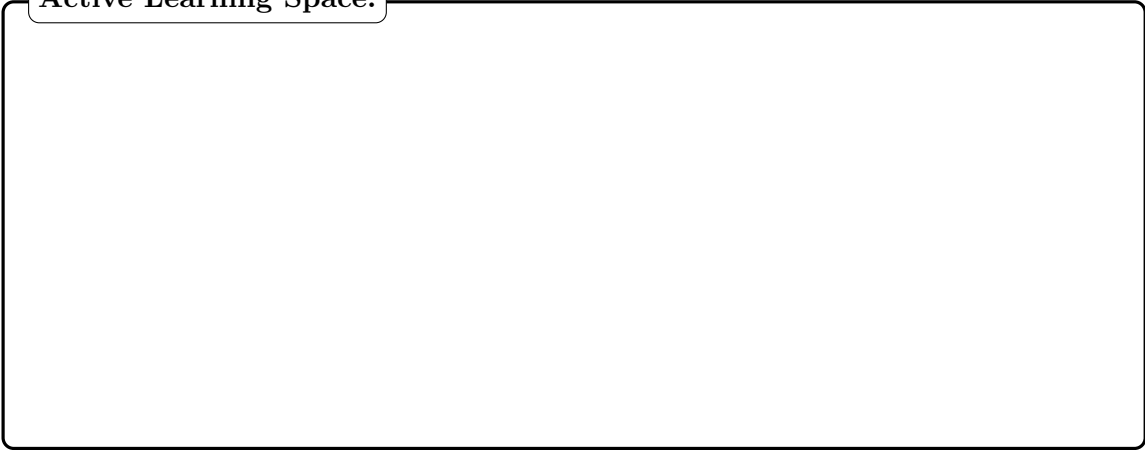
If $g(t)$ has Laplace transform $G(s)$ then the shifted function

$$u(t - a)g(t - a)$$

has Laplace transform

$$e^{-as}G(s).$$

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Example.

Determine the Laplace transform of

$$u(t - 4)[t - 4].$$

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Example.

Determine the inverse Laplace transform of

$$H(s) := \frac{1 - e^{-2s}}{s^2}.$$

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Example.

Determine the Laplace transform of

$$u(t - 4)[t - 4]^2.$$

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Example.

Determine the Laplace transform of

$$u(t - 2\pi) \sin(t - 2\pi).$$

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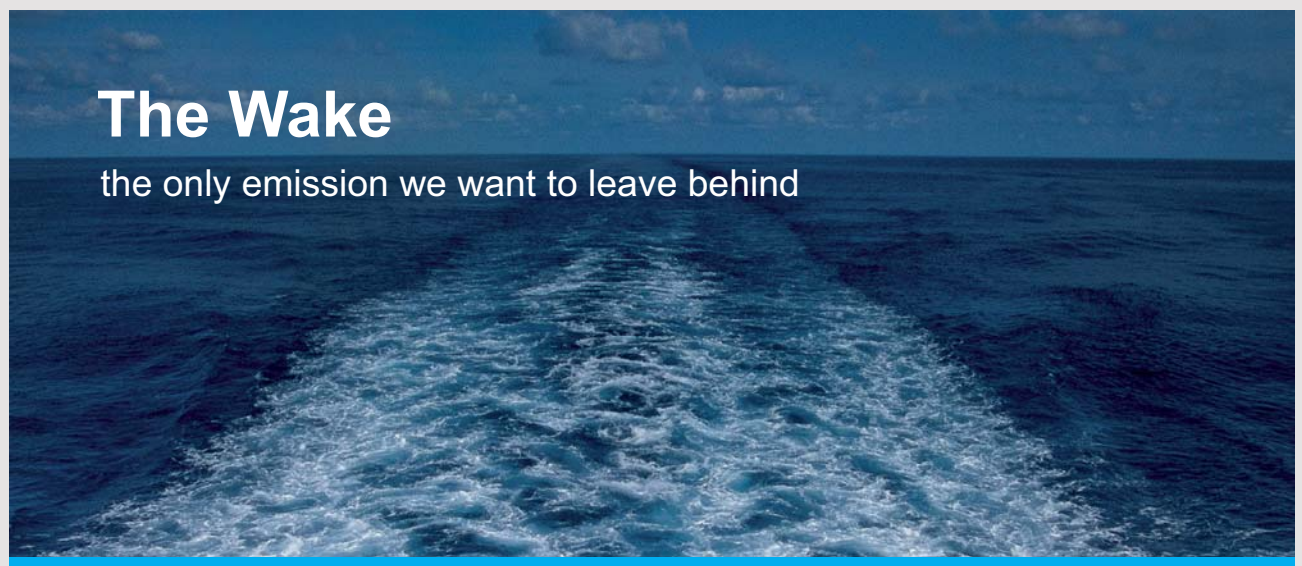
Example.

If

$$f(t) := \begin{cases} t, & \text{for } 0 \leq t < 4, \\ 2, & \text{for } t = 4 \\ 0, & \text{for } t > 4. \end{cases}$$

then find the Laplace transform of f .

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Example.

(Square wave example)

a) If $f(t) = f(t + T)$ for all $t > 0$ (i.e. f is periodic with period $T > 0$) then show

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt.$$

b) Hence compute $\mathcal{L}\{f(t)\}$, where $f(t + 2) = f(t)$ and

$$f(t) := \begin{cases} 1, & \text{for } 0 \leq t < 1, \\ 0, & \text{for } 1 < t < 2. \end{cases}$$

then find the Laplace transform of f .

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6.7 Transform of Derivatives

View this lesson on YouTube [36]

The primary use of Laplace transforms is to solve linear ODEs with constant coefficients. As a first step towards this, we now discuss how to take the transform of derivatives.

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\ &= sF(s) - f(0).\end{aligned}$$

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Hence,

$$\begin{aligned}\mathcal{L}\{f^{(n)}(t)\} &= s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0) \\ &= s^n F(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \dots - f^{(n-1)}(0).\end{aligned}$$

Example.

If

$$f(t) := \int_0^t g(\tau) d\tau$$

then show

$$F(s) = \mathcal{L}\{f(t)\} = \frac{G(s)}{s}.$$

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6.8 Solving IVPs with Laplace Transforms

We transform the IVP to an algebraic problem, solve the algebraic problem and then invert to obtain the solution to our original IVP.

The method will work for linear ODEs with constant coefficients. This method has an advantage over previous methods in the streamlining of initial conditions into the calculations.

The method is best illustrated via examples.

Example.

Use the Laplace transform to solve the higher-order IVP

$$y^{(iv)} - y = 0$$
$$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0.$$

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6.9 Applications of Laplace Transforms to PDE

View this lesson on YouTube [37]

- We will discover how to solve partial differential equations via applications of Laplace transforms.
- One benefit of this approach is its ability to streamline the treatment of PDE and associated boundary conditions.
- Such techniques are well suited to linear PDE with constant coefficients.

We define the Laplace transform of a piecewise continuous function $f = f(t)$ by

$$\mathcal{L}\{f(t)\} = F(s) := \int_0^{\infty} e^{-st} f(t) dt.$$

Now, if we have a function of two variables, $w(x, t)$ then

$$\mathcal{L}\{w(x, t)\} = W(x, s) := \int_0^{\infty} e^{-st} w(x, t) dt.$$

Important properties for transform methods to PDE will be the transform of partial derivatives:

$$\begin{aligned}\mathcal{L}\{w_t(x, t)\} &= s\mathcal{L}\{w(x, t)\} - w(x, 0) \\ \mathcal{L}\{w_{tt}(x, t)\} &= s^2\mathcal{L}\{w(x, t)\} - sw(x, 0) - w_t(x, 0) \\ \mathcal{L}\{w_x(x, t)\} &= \frac{\partial}{\partial x}\mathcal{L}\{w(x, t)\}\end{aligned}$$

The solution method is best illustrated by example.

Example.

Solve

$$xu_x - yu_y + u = 2.$$

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A large, empty rectangular box with a thin black border, intended for active learning or student work. The box is positioned below the 'Active Learning Space' header and occupies most of the page's width and height.

View this lesson on YouTube [38]

Example.

Solve, via Laplace transforms,

$$w_{tt} = w_{xx}, \quad w(0, t) = g(t), \quad \lim_{x \rightarrow \infty} w(x, t) = 0, \quad x, t \geq 0$$

where w is the deflection of a string, released from rest on the x -axis.

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View this lesson on YouTube [39]

Example.

a) Solve, using Laplace transforms:

$$T_t = T_{xx} \quad (6.9.1)$$

$$T(x, 0) = 0, \quad x \geq 0; \quad (6.9.2)$$

$$\lim_{x \rightarrow \infty} T(x, t) = 0; \quad \lim_{x \rightarrow \infty} T_x(x, t) = 0; \quad (6.9.3)$$

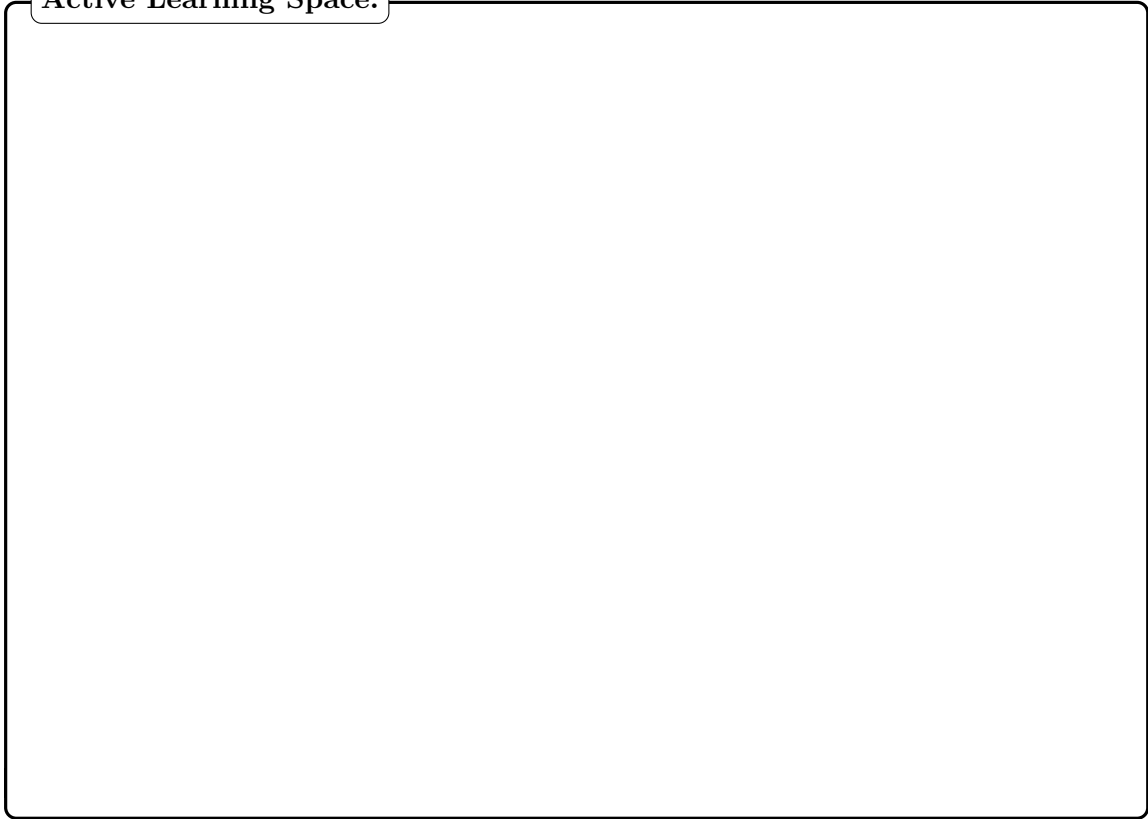
$$T(0, t) = 1, \quad t > 0. \quad (6.9.4)$$


b) Demonstrate that your answer above is a genuine solution to (6.9.1)–(6.9.4) by illustrating that your answer satisfies each of the above equations.

c) Will the method of separation of variables work for (6.9.1)–(6.9.4)? Explain.

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Chapter 7

Applications of Fourier Transforms to PDE

7.1 Introduction to Fourier Transforms

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- We will discover a new transform, known as the Fourier transform.
- Such ideas complement those of other integral transforms, such as Laplace transforms.
- Fourier transforms find useful applications in solving partial differential equations.
- Fourier transforms appear when limits are applied to Fourier series.
- In particular, Fourier transforms can accommodate problems with non-periodic functions, which Fourier series cannot do.

We define the Fourier transform of a piecewise continuous and absolutely integrable function $f = f(x)$ by

$$\mathcal{F}\{f(x)\} = \hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx} dx.$$

Playing the inverting role to our Fourier transform is the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(w)\} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)e^{iwx} dw.$$

Example.

Let $a > 0$. Compute the Fourier transform of

$$f(x) := \begin{cases} e^{-ax}, & \text{for } x \geq 0, \\ e^{ax}, & \text{for } x < 0. \end{cases}$$

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Example.

Let $a > 0$. Compute the Fourier transform of

$$f(x) := e^{-ax^2}.$$

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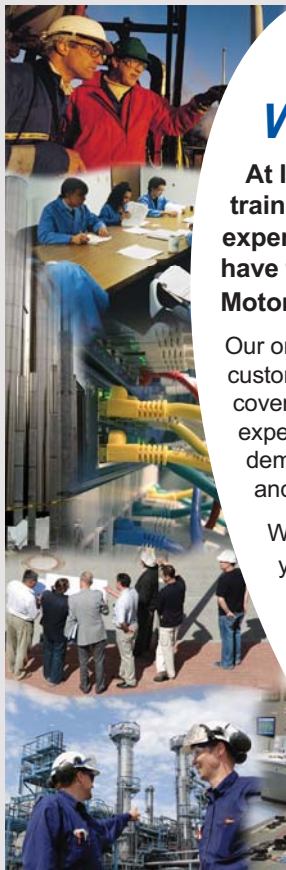
7.2 First Shifting Theorem of Fourier Transforms

View this lesson on YouTube [41]

If a is a constant and $\hat{f}(w) := \mathcal{F}\{f(x)\}$ then

$$\begin{aligned}\mathcal{F}\{f(x - a)\} &= e^{-iaw} \mathcal{F}\{f(x)\} \\ \mathcal{F}^{-1}\{e^{-iaw} \hat{f}(w)\} &= f(x - a).\end{aligned}$$

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Example.

Let c be a constant and consider

$$\hat{u}(w, t) := \hat{f}(w) \cos(cwt).$$

- a) Compute $u(x, t)$ via inverse Fourier transforms.
- b) Show that your result satisfies the wave equation $c^2 u_{xx} = u_{tt}$.

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View this lesson on YouTube [42]

Example.

Solve, via Fourier transforms,

$$\begin{aligned}u_{tt} &= u_{xx}, & -\infty < x < \infty, & \quad t > 0; \\u(x, 0) &= g(x), & -\infty < x < \infty; \\u_t(x, 0) &= 0, & -\infty < x < \infty; \\u(x, t) &\rightarrow 0, \quad u_x(x, t) \rightarrow 0; & \text{as } |x| \rightarrow \infty.\end{aligned}$$

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Important properties for Fourier transform methods to solve differential equations will be the transform of derivatives:

$$\begin{aligned}\mathcal{F}\{f'(x)\} &= iw\mathcal{F}\{f(x)\} \\ \mathcal{F}\{f''(x)\} &= -w^2\mathcal{F}\{f(x)\}\end{aligned}$$

where we have assumed $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, f' absolutely integrable on the x -axis, f continuous. In addition

$$\begin{aligned}\mathcal{F}\{u_x(x, t)\} &= iw\mathcal{F}\{u(x, t)\} \\ \mathcal{F}\{u_{xx}(x, t)\} &= -w^2\mathcal{F}\{u(x, t)\} \\ \mathcal{F}\{u_t(x, t)\} &= \frac{\partial}{\partial t}\mathcal{F}\{u(x, t)\}.\end{aligned}$$

The solution method for PDE is best illustrated by example.

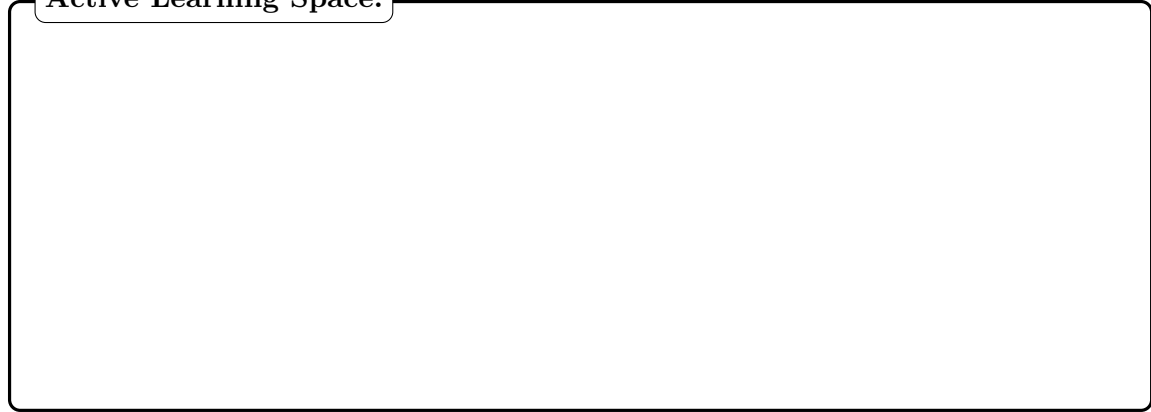
7.3 Second Shifting Theorem of Fourier Transforms

View this lesson on YouTube [43]
If a is a constant and $\hat{f}(w) := \mathcal{F}\{f(x)\}$ then

$$\mathcal{F}\{e^{iax} f(x)\} = \hat{f}(w - a)$$

$$\mathcal{F}^{-1}\{\hat{f}(w - a)\} = e^{iax} f(x).$$

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Example.

Compute

$$\mathcal{F} \left\{ e^{-(w-2)^2/4} \right\}$$

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7.4 Convolution Theorem of Fourier Transforms

View this lesson on YouTube [44]

Define the convolution of two functions f and g by

$$\begin{aligned}(f * g)(x) &:= \int_{-\infty}^{\infty} f(p)g(x-p) dp \\ &= \int_{-\infty}^{\infty} f(x-p)g(p) dp\end{aligned}$$

If f and g are piecewise continuous, bounded and absolutely integrable on the x -axis then

$$\mathcal{F}\{(f * g)(x)\} = \sqrt{2\pi}\mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}$$

$$\mathcal{F}^{-1}\{\hat{f}(w)\hat{g}(w)\} = \frac{1}{\sqrt{2\pi}}(f * g)(x).$$

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View this lesson on YouTube [45]

Example.

Solve, via Fourier transforms,

$$\begin{aligned}u_t &= u_{xx}, & -\infty < x < \infty, & \quad t > 0; \\u(x, 0) &= g(x), & -\infty < x < \infty; \\u(x, t) &\rightarrow 0, \quad u_x(x, t) \rightarrow 0; & \text{as } |x| \rightarrow \infty.\end{aligned}$$

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Chapter 8

Introduction to Green's Functions

8.1 Introduction

We will learn about

- Laplace's equation
- Green's identities.

If diffusion or wave-phenomena are independent of time (stationary) then $u_t = 0 = u_{tt}$ and the famous diffusion and wave equations reduce to what are known as Laplace's equations (in 1, 2 and 3 dimensions)

$$u_{xx} = 0 \tag{8.1.1}$$

$$\nabla \cdot \nabla u = \nabla^2 u = \Delta u := u_{xx} + u_{yy} = 0 \tag{8.1.2}$$

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{8.1.3}$$



A function u that satisfies Laplace's equation is called **harmonic**. The nonhomogeneous form of Laplace's equation is known as Poisson's equation

$$u_{xx} + u_{yy} = f$$

etc.

We shall examine the following problems

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in D \quad (8.1.4)$$

$$u(x, y) = g(x, y), \quad (x, y) \in \partial D \quad (8.1.5)$$

which is known as a Dirichlet problem, and

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in D \quad (8.1.6)$$

$$\nabla u \bullet \vec{n} = g(x, y), \quad (x, y) \in \partial D \quad (8.1.7)$$

which is known as a Neumann problem, where \vec{n} is the outward-pointing unit vector to D . The expression $\nabla u \bullet \vec{n}$ is sometimes denoted as $\partial u / \partial n$ and termed "the normal derivative".

8.2 Green's First Identity

View this lesson on YouTube [46]

Let D be a bounded region in the plane, whose boundary ∂D is a closed and piecewise smooth curve. If k and h are continuous, with continuous first and second order partial derivatives on D and ∂D , then

$$\oint_{\partial D} k \frac{\partial h}{\partial n} ds = \oint_{\partial D} k \nabla h \cdot \vec{n} ds = \iint_D [k \nabla^2 h + \nabla k \cdot \nabla h] dA.$$

The proof follows from the Green–Gauss divergence theorem

$$\oint_{\partial D} \vec{F} \cdot \vec{n} ds = \iint_D \nabla \cdot \vec{F} dA.$$

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8.3 Uniqueness for the Dirichlet Problem

View this lesson on YouTube [47]

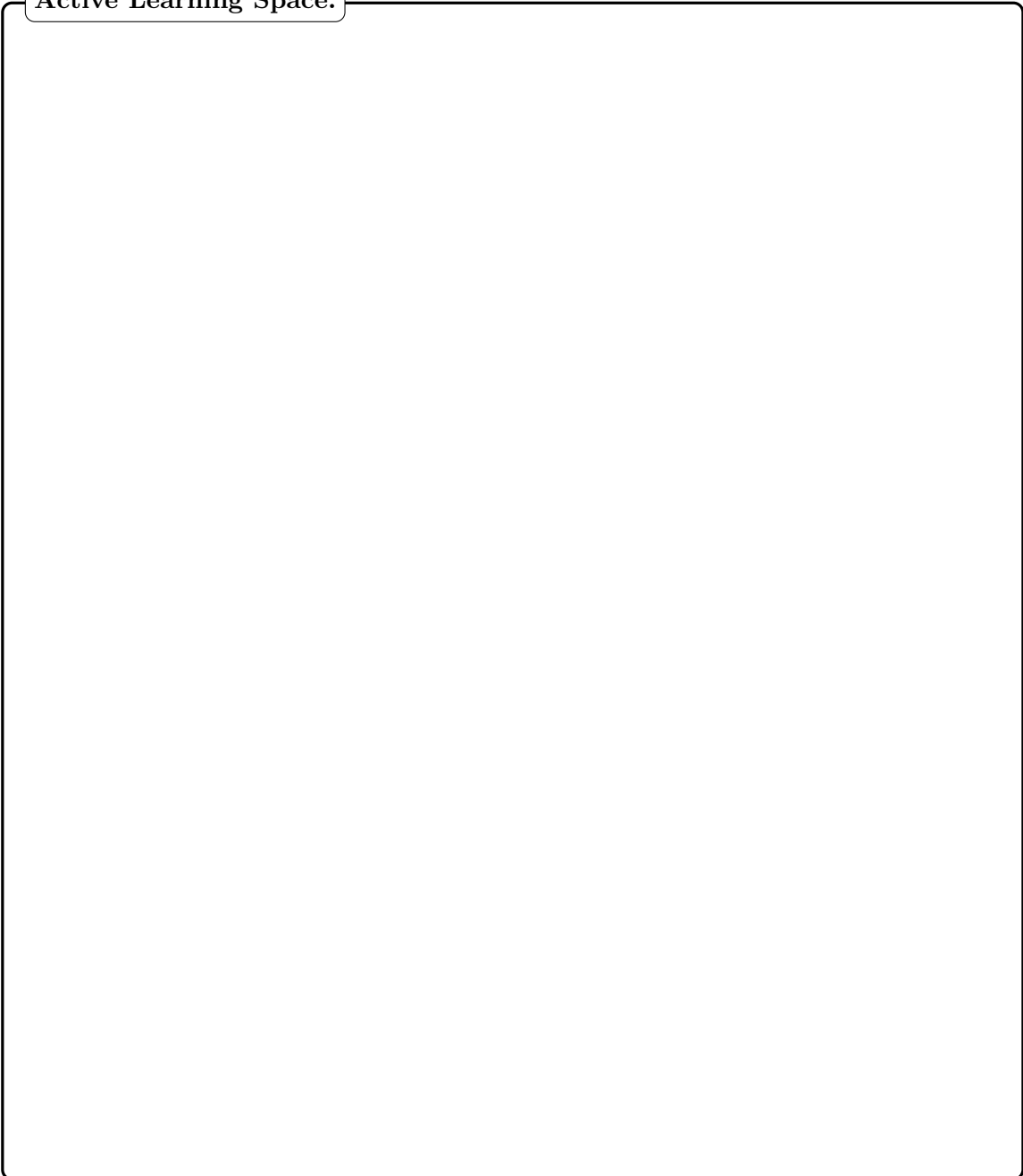
Consider the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in D \quad (8.3.1)$$

$$u(x, y) = g(x, y), \quad (x, y) \in \partial D. \quad (8.3.2)$$

The problem (8.3.1), (8.3.2) has, at most, one solution.

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8.4 Existence for the Dirichlet Problem

View this lesson on YouTube [48]

Consider the Neumann problem

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in D \quad (8.4.1)$$

$$\partial u / \partial n := \nabla u \bullet \vec{n} = g(x, y), \quad (x, y) \in \partial D. \quad (8.4.2)$$

A necessary condition for (8.4.1), (8.4.2) to have solution is

$$\oint_{\partial D} g(x, y) \, ds = 0. \quad (8.4.3)$$

The condition (8.4.3) can be interpreted in the following way. Suppose u is the steady-state temperature within the region D . Thus, $\partial u / \partial n = g$ is the flux of energy as heat across the boundary ∂D .

Condition (8.4.3) states that the net flux across ∂D must be zero to maintain a steady-state temperature.

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Example.

Let R denote the filled rectangle

$$R := \{(x, y) : 0 \leq x \leq 1; 0 \leq y \leq 1\}.$$

Does the following problem have a solution?

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & (x, y) \in R \\ \partial u / \partial n &:= \nabla u \bullet \vec{n} = g(x, y), & (x, y) \in \partial R \end{aligned}$$

where $g(0, y) = 3y^2$ on $0 \leq y \leq 1$; and $g \equiv 0$ on the remaining three sides?

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Chapter 9

Harmonic Functions and Maximum Principles

9.1 Introduction

View this lesson on YouTube [49]

We will learn about

- Mean value theorems
- Important inequalities and estimates
- Maximum principles

If diffusion or wave-phenomena are independent of time (stationary) then $u_t = 0 = u_{tt}$ and the famous diffusion and wave equations reduce to what are known as Laplace's equations (in 1, 2 and 3 dimensions)

$$u_{xx} = 0 \tag{9.1.1}$$

$$\nabla \cdot \nabla u = \nabla^2 u = \Delta u := u_{xx} + u_{yy} = 0 \tag{9.1.2}$$

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{9.1.3}$$

A function u that satisfies Laplace's equation is called **harmonic**. The nonhomogeneous form of Laplace's equation is known as Poisson's equation:

$$u_{xx} + u_{yy} = f$$

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9.2 First Mean Value Theorem for Harmonic Functions

Let u be a C^2 harmonic function in an open set $\Omega \subseteq \mathbb{R}^2$. If $B_r(a, b)$ is any ball with $\overline{B_r(a, b)} \subseteq \Omega$ then

$$u(a, b) = \frac{1}{2\pi r} \oint_{\partial B_r(a, b)} u(x, y) ds.$$

- The integral on the right-hand side is a (scalar) line integral.
- The theorem says that the value of a harmonic function u at a point (a, b) is equal to the average value of u over any circle around (a, b) .
- The proof utilizes the Gauss–Green theorem in the plane. If $\vec{F} = M(x, y)\vec{i} + N(x, y)\vec{j}$ then

$$\iint_{B_r(a, b)} \nabla \cdot \vec{F} dx dy = \oint_{\partial B_r(a, b)} M dy - N dx.$$

Proof. Let $\vec{F} = \nabla u$, so $M = u_x$ and $N = u_y$ and the Gauss–Green identity gives

$$\iint_{B_r(a, b)} \nabla \cdot \nabla u dx dy = \oint_{\partial B_r(a, b)} u_x dy - u_y dx.$$

Now, since $\Delta u = \nabla^2 u := \nabla \cdot \nabla u = 0$, we have

$$0 = \oint_{\partial B_r(a, b)} u_x dy - u_y dx.$$

Parametrize $\partial B_R(a, b)$ via

$$\vec{r}(t) = (x(t), y(t)) = (a + R \cos t, b + R \sin t), \quad 0 \leq t \leq 2\pi.$$

so that

$$\begin{aligned} \vec{r}'(t) &= (x'(t), y'(t)) = (-R \sin t, R \cos t), \quad 0 \leq t \leq 2\pi \\ \|\vec{r}'(t)\| &= R. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \oint_{\partial B_R(a,b)} u_x dy - u_y dx \\ &= \int_0^{2\pi} u_x(a + R \cos t, b + R \sin t) R \cos t \\ &\quad + u_y(a + R \cos t, b + R \sin t) R \sin t dt \\ &= \int_0^{2\pi} \frac{\partial}{\partial R} u(a + R \cos t, b + R \sin t) dt \\ &= \frac{d}{dR} \int_0^{2\pi} u(a + R \cos t, b + R \sin t) dt \end{aligned}$$

where we have used the chain rule and Leibniz rule for differentiating under integral signs.

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Integration from $R = 0$ to $R = r$ yields

$$\begin{aligned}
 0 &= \int_0^r \frac{d}{dR} \int_0^{2\pi} u(a + R \cos t, b + R \sin t) dt dR \\
 &= \int_0^{2\pi} u(a + r \cos t, b + r \sin t) dt - \int_0^{2\pi} u(a, b) dt \\
 &= \int_0^{2\pi} u(a + r \cos t, b + r \sin t) dt - 2\pi u(a, b) \\
 &= \frac{1}{r} \int_0^{2\pi} u(a + r \cos t, b + r \sin t) r dt - 2\pi u(a, b) \\
 &= \frac{1}{r} \oint_{\partial B_r(a,b)} u(x, y) ds - 2\pi u(a, b)
 \end{aligned}$$

and the result follows. □

- The proof of the first mean value theorem can be run backwards to form an “iff” type statement.
- If we replace “ u harmonic” with “ $\Delta u \geq 0$ ” (u subharmonic) then we obtain the estimate

$$u(a, b) \leq \frac{1}{2\pi r} \oint_{\partial B_r(a,b)} u(x, y) ds. \tag{9.2.1}$$

- A reversed inequality can be formed from $\Delta u \leq 0$ (u superharmonic).

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9.3 Maximum Principle for Subharmonic Functions

View this lesson on YouTube [50]

Important Idea.

Theorem: Let D be bounded and connected and

$$\Delta u(x, y) \geq 0, \quad \text{for all } (x, y) \in D.$$

If u attains its maximum value M on \bar{D} at any point in D , then $u \equiv M$.

Proof. Let u attain its max value M at $(x_0, y_0) \in D$. Now, since $u \leq M$ and $u(x_0, y_0) = M$ then it follows from (9.2.1) that for each circle ∂B_r we have

$$M = u(x_0, y_0) \leq \frac{1}{2\pi r} \oint_{\partial B_r(a,b)} u(x, y) ds \leq M$$

and so $u \equiv M$ on every circle centred at (x_0, y_0) and lying in D .

Now let there be a point $(x_1, y_1) \in D$ such that $u(x_1, y_1) < M$. Then, by continuity, the same inequality holds in a neighbourhood of (x_1, y_1) .

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Connect the points (x_0, y_0) and (x_1, y_1) by a curve lying in D . Let (x_2, y_2) be the first point where $u(x_2, y_2) = M$. But then u is not identically equal to M on any sufficiently small circle centred at (x_2, y_2) .

This contradiction proves the result. \square

Now, if D is bounded and u is continuous on $D \cup \partial D$ with u non-constant then the values of u in D are below the maximum value of u on the boundary ∂D .

Important Idea.

That is, every non-constant subharmonic function on a bounded set D must achieve its maximum value on the boundary of D .

There is also a minimum principle for superharmonic functions ($\Delta u \leq 0$). Thus, if we combine these ideas we see that

Important Idea.

Every nonconstant harmonic function u must attain its maximum and minimum values on the boundary.



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9.4 Independent Learning - More Properties of Harmonic Functions

The Meaning of Subharmonic Functions

Suppose that u is subharmonic ($\Delta u \geq 0$) and v is harmonic ($\Delta v = 0$) in a region D with boundary ∂D . The function

$$w := u - v$$

will be subharmonic on D , ie

$$\Delta w \geq 0.$$

If $u \equiv v$ on ∂D then

$$w \equiv 0, \quad \text{on } \partial D$$

and so, by the maximum principle,

$$w \leq 0, \quad \text{on } D.$$

Thus

$$u \leq v, \quad \text{on } D.$$

Second mean value theorem for harmonic functions

Let u be a C^2 harmonic function in an open set $\Omega \subseteq \mathbb{R}^2$. If $B_r(a, b)$ is any ball with $\overline{B_r(a, b)} \subseteq \Omega$ then

$$u(a, b) = \frac{1}{\pi r^2} \iint_{B_r(a, b)} u(x, y) \, dx \, dy.$$

Gradient estimate for non-negative harmonic functions

If u is harmonic ($\Delta u = 0$) and non-negative on D then for $\overline{B_R(a, b)} \subseteq D$ we have

$$\|\nabla u(a, b)\| \leq \frac{2\sqrt{2}}{r} u(a, b).$$

Proof. It can be shown that if u is harmonic, then u_x and u_y are also harmonic. Hence by the second MVT applied to u_x we have

$$u_x(a, b) = \frac{1}{\pi r^2} \iint_{B_r(a, b)} u_x(x, y) \, dx \, dy.$$

Now, we can apply the Green–Gauss theorem (with $\vec{F} = u\vec{i}$) to the double integral to obtain

$$\begin{aligned} \frac{1}{\pi r^2} \iint_{B_r(a, b)} u_x(x, y) \, dx \, dy &= \frac{1}{\pi r^2} \oint_{\partial B_r(a, b)} u(x, y) \, dy \\ &= \frac{1}{\pi r^2} \int_0^{2\pi} u(a + r \cos t, b + r \sin t) \, r \cos t \, dt. \end{aligned}$$

Now, noting that u is non-negative, we have

$$\begin{aligned} |u_x(a, b)| &\leq \frac{1}{\pi r^2} \oint_{\partial B_r(a, b)} u(x, y) dy \\ &\leq \frac{1}{\pi r^2} \int_0^{2\pi} u(a + r \cos t, b + r \sin t) r dt \\ &= \frac{1}{\pi r^2} \oint_{\partial B_r(a, b)} u(x, y) ds. \end{aligned}$$

Hence, the first MVT yields

$$\begin{aligned} |u_x(a, b)| &\leq \frac{1}{\pi r^2} 2\pi r u(a, b) \\ &= \frac{2}{r} u(a, b). \end{aligned}$$

Similarly, $|u_y(a, b)| \leq ru(a, b)/2$. Hence, $\|\nabla u(a, b)\| \leq 2\sqrt{2}u(a, b)/r$.

Harmonic and bounded functions are constant

Important Idea.

Let u be a harmonic function on the whole plane \mathbb{R}^2 . If u is bounded above or below, then u must be a constant.

Proof. Let M be a constant such that $u \leq M$ on \mathbb{R}^2 . Let $v := M - u$ and it follows that v is harmonic and non-negative on \mathbb{R}^2 .

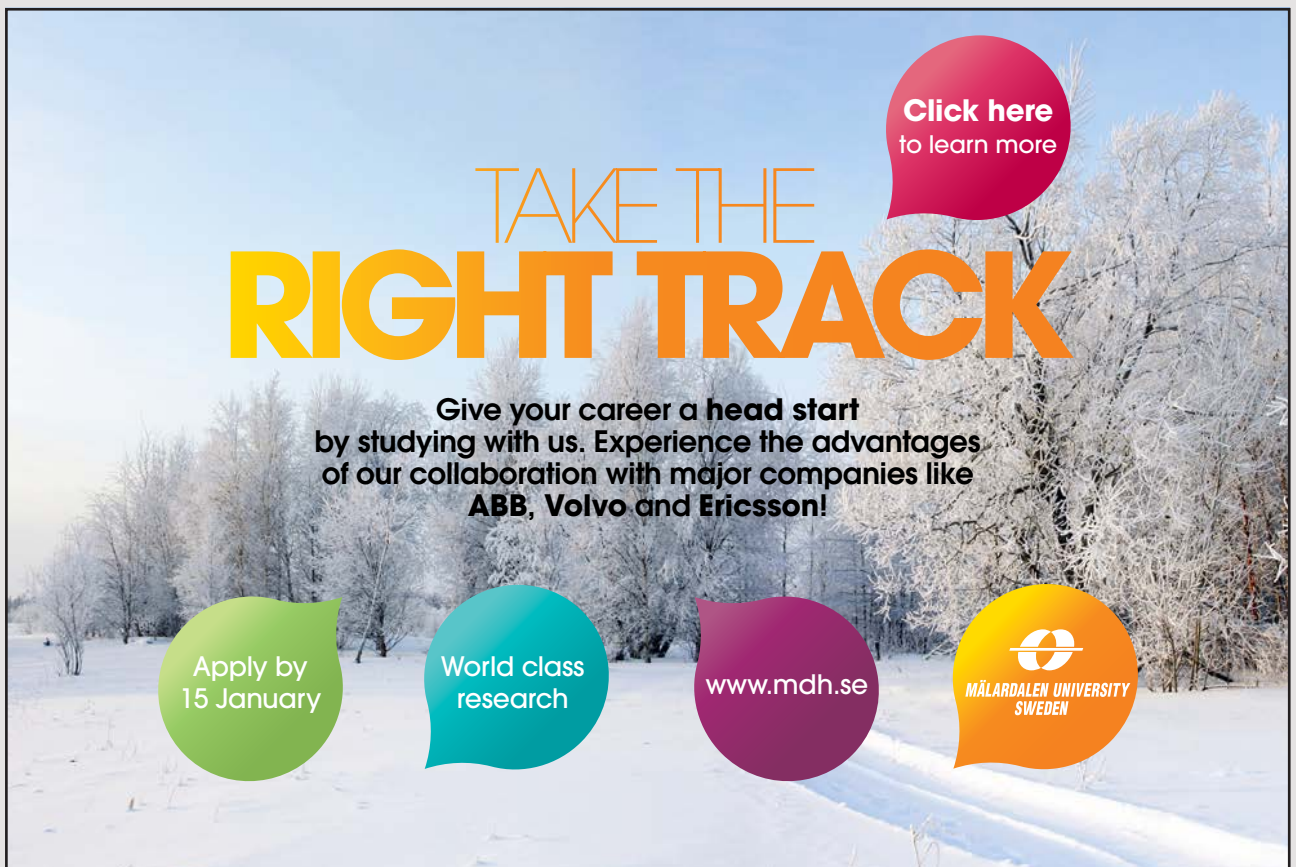
Thus, for any point (a, b) , the gradient estimate gives

$$\|\nabla v(a, b)\| \leq \frac{2\sqrt{2}}{r}v(a, b)$$

ie

$$\|\nabla u(a, b)\| \leq \frac{2\sqrt{2}}{r}(M - u(a, b))$$

for all $r > 0$. If we let $r \rightarrow \infty$ then we obtain $\nabla u(a, b) = \vec{0}$, so that u is a constant. \square



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