1 Integral Equations and Picard's Method

1.1 Integral equations and their relationship to differential equations

Four main types of integral equations will appear in this book: their names occur in the table below. Suppose that $f : [a, b] \to \mathbb{R}$ and $K : [a, b]^2 \to \mathbb{R}$ are continuous, and that λ, a, b are constants.

Volterra non-homogeneous	$y(x) = f(x) + \int_a^x K(x,t)y(t)\,dt$
Volterra homogeneous	$y(x) = \int_a^x K(x,t)y(t)\,dt$
Fredholm non-homogeneous	$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)\,dt$
Fredholm homogeneous	$y(x) = \lambda \int_a^b K(x,t)y(t)\,dt$

where $x \in [a, b]$. Note that the Volterra equation can be considered as a special case of the Fredholm equation when $K(x, t) = 0$ for $t > x$ in $[a, b]$.

We will search for continuous solutions $y = y(x)$ to such equations. On occasion, x may range over a different domain from $[a, b]$; in which case, the domains of f and K will need appropriate modification.

The function $K = K(x,t)$ appearing in all four equations is called the *kernel* of the integral equation. Such a kernel is *symmetric* if $K(x,t) = K(t,x)$, for all $x, t \in [a,b]$.

A value of the constant λ, for which the homogeneous Fredholm equation has a solution $y = y(x)$ which is *not* identically zero on $[a,b]$, is called an *eigenvalue*, or *characteristic value*, of that equation, and such a non-zero solution $y = y(x)$ is called an *eigenfunction*, or *characteristic function*, 'corresponding to the eigenvalue λ'. The analogy with linear algebra is not accidental, as will be apparent in later chapters.

To investigate the relationship between integral and differential equations, we will need the following lemma which will allow us to replace a double integral by a single one.

Lemma 1 (Replacement Lemma) Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Then

$$\int_a^x \int_a^{x'} f(t)\,dt dx' \;=\; \int_a^x (x-t)f(t)\,dt, \qquad (x \in [a,b]).$$

Proof Define $F : [a,b] \to \mathbb{R}$ by

$$F(x) \;=\; \int_a^x (x-t)f(t)\,dt, \qquad (x \in [a,b]).$$

As $(x-t)f(t)$ and $\dfrac{\partial}{\partial x}[(x-t)f(t)]$ are continuous for all x and t in $[a,b]$, we can use [G] of Chapter 0 to differentiate F:

$$F'(x) \;=\; [(x-t)f(t)]\Big|_{t=x} \frac{d}{dx}x + \int_a^x \frac{\partial}{\partial x}\left[(x-t)f(t)\right]dt \;=\; \int_a^x f(t)\,dt.$$

Since, again by [G] of Chapter 0, $\displaystyle\int_a^x f(t)\,dt$, and hence $\dfrac{dF}{dx}$, are continuous functions of x on $[a,b]$, we may now apply the Fundamental Theorem of Calculus ([E] of Chapter 0) to deduce that

$$F(x') \;=\; F(x') - F(a) \;=\; \int_a^{x'} F'(x)\,dx \;=\; \int_a^{x'}\int_a^x f(t)\,dtdx.$$

Swapping the roles of x and x', we have the result as stated. □

Alternatively, define, for $(t, x') \in [a, x]^2$,

$$G(t, x') \;=\; \begin{cases} f(t) & \text{when } a \le t \le x' \le x, \\ 0 & \text{when } a \le x' \le t \le x. \end{cases}$$

The function $G = G(t, x')$ is continuous, except on the line given by $t = x'$, and hence integrable. Using Fubini's Theorem ([F] of Chapter 0),

$$\begin{aligned} \int_a^x \int_a^{x'} f(t)\, dt dx' &= \int_a^x \left(\int_a^x G(t, x')\, dt \right) dx' \\ &= \int_a^x \left(\int_a^x G(t, x')\, dx' \right) dt \\ &= \int_a^x \left(\int_t^x f(t)\, dx' \right) dt \\ &= \int_a^x (x - t) f(t)\, dt. \end{aligned}$$

□

We now give an example to show how Volterra and Fredholm integral equations can arise from a single differential equation (as we shall see, depending on which sort of conditions are applied at the boundary of the domain of its solution).

Example 1 Consider the differential equation

$$y'' + \lambda y \;=\; g(x), \quad (x \in [0, L]),$$

where λ is a positive constant and g is continuous on $[0, L]$. (Many readers will already be able to provide a method of solution. However, what we are considering here is equivalent formulations in terms of integral equations.) Integration from 0 to x ($x \in [0, L]$) gives

$$y'(x) - y'(0) + \lambda \int_0^x y(t)\, dt \;=\; \int_0^x g(t)\, dt.$$

(Note that, as y'' must exist for any solution y, both y and $y'' = g(x) - \lambda y$ are continuous, so that $\int_0^x y''(t)\,dt = y'(x) - y'(0)$ by [E] of Chapter 0.) As $y'(0)$ is a constant, a further integration from 0 to x and use of the Replacement Lemma twice now gives

$$y(x) - y(0) - xy'(0) + \lambda \int_0^x (x-t)y(t)\,dt \;=\; \int_0^x (x-t)g(t)\,dt. \tag{1}$$

At this point comes the parting of the ways: we consider two ways in which conditions can be applied at the boundary of the domain of a solution.

(i) **Initial conditions** where y and y' are given at the 'initial' point. Suppose here that $y(0) = 0$ and $y'(0) = A$, a given real constant. Then

$$y(x) \;=\; Ax + \int_0^x (x-t)g(t)\,dt - \lambda \int_0^x (x-t)y(t)\,dt. \tag{2}$$

Thus we have a Volterra non-homogeneous integral equation with, in the notation of the above table,

$$K(x,t) \;=\; \lambda(t-x),$$
$$f(x) \;=\; Ax + \int_0^x (x-t)g(t)\,dt,$$

which becomes homogeneous if and only if A and g satisfy

$$Ax + \int_0^x (x-t)g(t)\,dt \;=\; 0.$$

All equations are valid for x in $[0, L]$.

(ii) **Boundary conditions** where y is given at the end-points of an interval. Suppose here that $y(0) = 0$ and $y(L) = B$, another given constant. Then, putting $x = L$ in (1), we have

$$y'(0) \;=\; \frac{1}{L}\left(\lambda \int_0^L (L-t)y(t)\,dt - \int_0^L (L-t)g(t)\,dt + B\right). \tag{3}$$

Substituting back into (1) and writing, for appropriate h,

$$\int_0^L h(t)\,dt \;=\; \int_0^x h(t)\,dt + \int_x^L h(t)\,dt,$$

one easily derives (and it is an exercise for the reader to check that)

$$(4) \qquad y(x) \;=\; f(x) + \lambda \int_0^L K(x,t)y(t)\,dt \qquad (x \in [0, L])$$

where

$$K(x,t) \;=\; \begin{cases} \dfrac{t}{L}\,(L-x) & \text{when} \quad 0 \le t \le x \le L \\[2ex] \dfrac{x}{L}\,(L-t) & \text{when} \quad 0 \le x \le t \le L \end{cases}$$

and

$$f(x) \;=\; \frac{Bx}{L} - \int_0^L K(x,t)g(t)\,dt.$$

This time we have a non-homogeneous Fredholm equation (which becomes homogeneous when $f = 0$ on $[0, L]$). We will come across this type of kernel again in our discussion of Green's functions: note that the form of $K(x, t)$ 'changes' along the line $x = t$.

It is *important* to notice that, not only can the original differential equation be recovered from the integral equations (2), (4) by differentiation, but that, so can the initial and boundary conditions. Demonstration of these facts is left as exercises. □

Exercise 1 Recover $y'' + \lambda y = g(x),\ y(0) = 0,\ y'(0) = A$ from (2), using differentiation and [G] of Chapter 0.

Exercise 2 Solve the integral equation

$$y(x) \;=\; e^x + 4\int_0^x (x-t)y(t)\,dt$$

by first converting it to a differential equation with appropriate initial conditions.

Exercise 3 Suppose that p is a continuously differentiable function, nowhere zero on $[a, b]$, and define

$$P(x) \;=\; \int_a^x \frac{dt}{p(t)}, \qquad (x \in [a, b]).$$

Show that a solution of the differential equation

$$\frac{d}{dx}\left(p(x)y'\right) = q(x)y + g(x),$$

(where q and g are continuous functions on $[a, b]$), with initial conditions $y(a) = A$, $y'(a) = B$, satisfies the Volterra integral equation

$$y(x) = f(x) + \int_a^x K(x,t)y(t)\,dt, \qquad (x \in [a,b]),$$

where

$$K(x,t) = q(t)(P(x) - P(t)),$$

and

$$f(x) = A + Bp(a)P(x) + \int_a^x (P(x) - P(t))g(t)\,dt.$$

Deduce that a solution of the equation

$$xy'' - y' - x^2 y = 8x^3,$$

with initial conditions $y(1) = 1$, $y'(1) = 4$, satisfies the equation

$$y(x) = x^4 + \tfrac{1}{2}\int_1^x (x^2 - t^2)y(t)\,dt, \qquad (x \geq 1).$$

Exercise 4 Find all the continuous eigenfunctions and the corresponding eigenvalues of the homogeneous Fredholm equation

$$y(x) = \lambda \int_0^1 K(x,t)y(t)\,dt,$$

where

$$K(x,t) = \begin{cases} x(1-t) & \text{when} \quad 0 \leq x \leq t \leq 1 \\ t(1-x) & \text{when} \quad 0 \leq t \leq x \leq 1 \end{cases}$$

by first converting it to a differential equation with appropriate boundary conditions.

Exercise 5 The thrice continuously differentiable real-valued function $y = y(x)$ satisfies the differential equation $y''' = f$ and is subject to the conditions $y(0) = y(1) = y(2) = 0$. By performing three integrations, show that a solution of the equation may be written,

$$y(x) = \int_0^2 L(x,t)f(t)\,dt,$$

for appropriate $L(x,t)$. You should determine such an $L(x,t)$.

1.2 Picard's method

In this section, we shall describe Picard's method, as used in the construction of a unique solution of an integral equation. This involves the construction of a sequence (y_n) of functions and, correspondingly, an infinite series Σu_n, where each u_n is defined by

$$u_n \;=\; y_n - y_{n-1}, \qquad (n = 1, 2, \ldots).$$

The Weierstrass M-test ([H] of Chapter 0) is used to show that the series is uniformly convergent to a function u. But notice that the N-th partial sum of the series is

$$\sum_{n=1}^{N}(y_n - y_{n-1}) \;=\; y_N - y_0.$$

So, the sequence (y_n) is uniformly convergent to $u + y_0$, which turns out (using the uniform convergence) to be the (unique) solution of the given integral equation.

We shall now put some clothes on this bare model by considering the Volterra integral equation,

$$y(x) \;=\; f(x) + \int_a^x K(x,t)y(t)\,dt, \qquad (x \in [a,b])$$

where f is continuous on $[a,b]$ and K and $\dfrac{\partial K}{\partial x}$ are continuous on $[a,b]^2$ (it is actually sufficient for K and $\dfrac{\partial K}{\partial x}$ to be continuous on the triangular region $\{(x,t) : a \le t \le x \le b\}$). We show that the integral equation has a unique continuous solution.

Inductively, we first define the sequence (y_n) 'by iteration': put

(5) $$y_0(x) \;=\; f(x), \qquad (x \in [a,b])$$

which is continuous by hypothesis, and suppose that y_k $(1 \le k \le n-1)$ has been defined as a continuous function on $[a,b]$ by the formula

$$y_k(x) \;=\; f(x) + \int_a^x K(x,t)y_{k-1}(t)\,dt.$$

Then for each x in $[a,b]$, $K(x,t)y_{n-1}(t)$ is a continuous, and hence integrable, function of t on $[a,x]$. So, we may define

(6) $$y_n(x) \;=\; f(x) + \int_a^x K(x,t)y_{n-1}(t)\,dt, \qquad (x \in [a,b]).$$

By [G] of Chapter 0, the integral in (6) is a differentiable, and therefore continuous, function of x on $[a,b]$ and thus an inductive definition of the sequence (y_n) of functions continuous on $[a,b]$ via formulas (5) and (6) is complete. Second, we find non-negative constants M_n such that

$$|u_n(x)| \;=\; |y_n(x) - y_{n-1}(x)| \;\leq\; M_n,$$

for all $n \in \mathbb{N}$ and all $x \in [a,b]$, with

$$\sum_{n=1}^{\infty} M_n \text{ convergent.}$$

Again we proceed by induction. We start by noting that, as K and f are continuous functions defined on, respectively, $[a,b]^2$ and $[a,b]$, K and f are bounded (by [A] of Chapter 0). Suppose that

$$|K(x,t)| \;\leq\; L, \quad |f(x)| \;\leq\; M \quad \text{for all } x,t \in [a,b],$$

where L, M are non-negative constants. For the first step in the induction, we have

$$\begin{aligned} |y_1(x) - y_0(x)| \;&=\; \left| \int_a^x K(x,t) y_0(t)\, dt \right| \\ &\leq\; \int_a^x |K(x,t)||y_0(t)|\, dt \qquad \text{(by [D] of Chapter 0)} \\ &\leq\; LM(x-a) \end{aligned}$$

for all x in $[a,b]$. For an inductive hypothesis, we take

$$(7) \qquad |y_{n-1}(x) - y_{n-2}(x)| \;\leq\; L^{n-1} M \frac{(x-a)^{n-1}}{(n-1)!}, \quad \text{for all } x \in [a,b],$$

where $n \geq 2$. (The curious reader may wonder how one might *ab initio* strike on such a hypothesis: he is referred to Exercise 6 below.)

Then, again using [D] of Chapter 0,

$$\begin{aligned} |y_n(x) - y_{n-1}(x)| &= \left| \int_a^x K(x,t)\{y_{n-1}(t) - y_{n-2}(t)\}\, dt \right| \\ &\le \int_a^x |K(x,t)||y_{n-1}(t) - y_{n-2}(t)|\, dt \\ &\le \int_a^x L.L^{n-1} M \frac{(t-a)^{n-1}}{(n-1)!}\, dt \\ &= L^n M \frac{(x-a)^n}{n!} \end{aligned}$$

for all x in $[a,b]$. One should note that, in the middle of this argument, one substitutes a bound for $|y_{n-1} - y_{n-2}|$ as a function of t and not of x. This is what gives rise to the 'exponential term' $(x-a)^n/n!$ (As will be discovered below, the fixed limits of integration in the analogous Fredholm equation give rise to a term of a geometric series.) Having inductively found bounds for all the $|y_n - y_{n-1}|$ over $[a,b]$ we can now define the non-negative constants M_n as follows:

$$|y_n(x) - y_{n-1}(x)| \le L^n M \frac{(x-a)^n}{n!} \le L^n M \frac{(b-a)^n}{n!} \equiv M_n$$

for $n \ge 1$. Now,

$$\sum_{n=1}^{\infty} M_n = M \sum_{n=1}^{\infty} \frac{\{L(b-a)\}^n}{n!} = M(e^{L(b-a)} - 1),$$

the exponential series for e^x being convergent for all values of its argument x. So, all the hypotheses for the application of the Weierstrass M-test ([H] of Chapter 0) are satisfied and we can deduce that

$$\sum_{n=1}^{\infty} (y_n - y_{n-1})$$

is uniformly convergent on $[a,b]$, to $u : [a,b] \to \mathbb{R}$, say. Then, as we showed above in our general discussion, the sequence (y_n) converges uniformly to $y \equiv u + y_0$ on $[a,b]$, which must be *continuous* on $[a, b]$, since every y_n is (use [I](a) of Chapter 0). Hence, given $\varepsilon > 0$, there exists N such that

$$|y(x) - y_n(x)| \ < \ \varepsilon, \quad \text{for all } n \geq N \text{ and all } x \in [a, b].$$

So,

$$|K(x,t)y(t) - K(x,t)y_n(t)| \ \leq \ L\varepsilon, \quad \text{for all } n \geq N \text{ and all } x, t \in [a, b].$$

So, the sequence $(K(x,t)y_n(t))$ converges uniformly, as a function of t, to $K(x,t)y(t)$. Therefore, by [I](a) of Chapter 0,

$$\int_a^x K(x,t)y_n(t)\,dt \quad \text{converges to} \quad \int_a^x K(x,t)y(t)\,dt.$$

Letting n tend to infinity in (6), we have shown the existence of a continuous solution $y = y(x)$ of the given integral equation.

To proceed to a proof of uniqueness of the continuous solution, we suppose that there exists another such solution $Y = Y(x)$. The continuous function $y - Y$ is bounded on $[a, b]$ (by [A] of Chapter 0). Suppose that

$$|y(x) - Y(x)| \ \leq \ P, \quad \text{for all } \ x \in [a, b],$$

where P is a non-negative constant, Then, as both y and Y satisfy the integral equation,

$$\begin{aligned} |y(x) - Y(x)| \ &= \ \left| \int_a^x K(x,t)(y(t) - Y(t))\,dt \right| \\ &\leq \ \int_a^x |K(x,t)||y(t) - Y(t)|\,dt \\ &\leq \ LP(x - a), \qquad \text{for all } \ x \in [a, b]. \end{aligned}$$

Inductively, suppose that

$$|y(x) - Y(x)| \ \leq \ L^{n-1}P\frac{(x-a)}{(n-1)!}^{n-1}, \qquad \text{for all } x \in [a, b] \text{ and } n \geq 2.$$

Then,

$$\begin{aligned} |y(x) - Y(x)| &= \left| \int_a^x K(x,t)(y(t) - Y(t))\, dt \right| \\ &\leq \int_a^x L.L^{n-1} P \, \frac{(t-a)^{n-1}}{(n-1)!}\, dt \\ &= L^n P \frac{(x-a)^n}{n!}, \qquad \text{for all } x \in [a,b]. \end{aligned}$$

So,

$$|y(x) - Y(x)| \leq L^n P \frac{(b-a)^n}{n!}, \qquad \text{for all } x \in [a,b] \text{ and } n = 1, 2, \ldots$$

and, since the right hand side of the inequality tends to 0 as $n \to \infty$,

$$y(x) = Y(x), \qquad \text{for all } x \in [a,b].$$

□

Digression on existence and uniqueness

The reader may wonder about the need or usefulness of such results as the above which establish the existence and uniqueness of a solution of, in this case, an integral equation. Why will a list of methods for solving particular kinds of equations, as a student is often encouraged to acquire, not suffice? The fact is that a number of equations, quite simple in form, do not possess solutions at all. An existence theorem can ensure that the seeker's search for a solution may not be fruitless. Sometimes one solution of a given equation is easy to find. Then a uniqueness theorem can ensure that the success so far achieved is complete, and no further search is needed.

One further word on the proof of existence theorems is in order here. There is no reason why such a proof should indicate any way in which one can actually find a solution to a given equation, and it often does not. However, many existence proofs do actually provide a recipe for obtaining solutions. The proof above does in fact provide a useful method which the reader should employ in completing Exercise 7 below.

Exercise 6 Calculate bounds for $|y_2(x) - y_1(x)|$ and $|y_3(x) - y_2(x)|$ to convince yourself of the reasonableness of the inductive hypothesis (7) in the above proof.

Exercise 7 Find the Volterra integral equation satisfied by the solution of the differential equation

$$y'' + xy = 1,$$

with initial conditions $y(0) = y'(0) = 0$. Use the above iterative method as applied to this integral equation to show that the first two terms in a convergent series expansion for this solution are

$$\frac{1}{2}\,x^2 - \frac{1}{40}\,x^5.$$

Be careful to prove that no other term in the expansion will contain a power of x less than or equal to 5.

Picard's method can also be applied to the solution of the Fredholm equation

$$y(x) \;=\; f(x) + \lambda \int_a^b K(x,t)y(t)\,dt, \tag{8}$$

where f is continuous on $[a,b]$ and K continuous on $[a,b]^2$. On this occasion, the iterative procedure

$$\begin{aligned} y_0(x) \;&=\; f(x) \\ y_n(x) \;&=\; f(x) + \lambda \int_a^b K(x,t)y_{n-1}(t)\,dt, \qquad (n \geq 1) \end{aligned} \tag{9}$$

for all x in $[a,b]$, gives rise (as the reader is asked to check) to the bound-inequalities

$$|y_n(x) - y_{n-1}(x)| \;\leq\; |\lambda|^n L^n M(b-a)^n \;\equiv\; M_n$$

for all $n \geq 1$ and all $x \in [a,b]$, where $|K| \leq L$, $|f| \leq M$, say. The series

$$\sum_{n=1}^{\infty} M_n$$

is geometric with common ratio $|\lambda|L(b-a)$, and so converges if $|\lambda L(b-a)| < 1$, that is, if

$$|\lambda| \;<\; \frac{1}{L(b-a)}\,, \tag{10}$$

assuming $L > 0$ and $b > a$, strictly. This additional sufficient condition ensures that a solution to the integral equation exists. The details of the remainder of the proof of existence and uniqueness here parallel those for the Volterra equation, and the reader is asked to supply them and note the differences.

No claim has been made above that the bound given for $|\lambda|$ is the best, that is, the largest, to ensure existence of a solution. We shall return to this point later during our discussion of the Fredholm Alternative.

Exercise 8 Suppose that

$$y(x) = 1 + \int_0^x e^{-t/x} y(t)\,dt$$

for $x > 0$ and that $y(0) = 1$. Show that the sequence $(y_n)_{n\geq 0}$ produced by Picard iteration is given by

$$\begin{aligned} y_0(x) &= 1 \\ y_n(x) &= 1 + \sum_{j=1}^{n} a_j x^j, \qquad (n \geq 1) \end{aligned}$$

for $x \geq 0$, where

$$a_1 = \int_0^1 e^{-s}\,ds$$

and

$$a_n = a_{n-1} \int_0^1 s^{n-1} e^{-s}\,ds \qquad (n \geq 2).$$

Exercise 9 For each of the following Fredholm equations, calculate the sequence $(y_n)_{n\geq 0}$ produced by Picard iteration and the bound on $|\lambda|$ for which the sequence converges to a solution (which should be determined) of the integral equation. Compare this bound with the bound given by the inequality (10) above.

(a) $$y(x) = x^2 + \lambda \int_0^1 x^2 t y(t)\,dt \qquad (x \in [0,1])$$

(b) $$y(x) = \sin x + \lambda \int_0^{2\pi} \sin(x+t) y(t)\,dt \qquad (x \in [0, 2\pi])$$

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2 Existence and Uniqueness

We begin this chapter by asking the reader to review Picard's method introduced in Chapter 1, with particular reference to its application to Volterra equations. This done, we may fairly speedily reach the essential core of the theory of ordinary differential equations: existence and uniqueness theorems.

To see that this work is essential, we need go no further than consider some very simple problems. For example, the problem of finding a differentiable function $y = y(x)$ satisfying

$$y' \;=\; y^2 \quad \text{with} \quad y(0) = 1$$

has the solution $y = (1-x)^{-1}$ which does not exist at $x = 1$; in fact, it tends to infinity ('blows up') as x tends to 1.

On the other hand, there are (Exercise 1) an infinite number of solutions of

$$y' \;=\; 3y^{\frac{2}{3}} \quad \text{with} \quad y(0) = 0$$

of which $y = 0$ identically and $y = x^3$ are the most obvious examples.

Further, most differential equations *cannot* be solved by performing a sequence of integrations, involving only 'elementary functions': polynomials, rational functions, trigonometric functions, exponentials and logarithms. The celebrated equation of Riccati,

$$y' \;=\; 1 + xy^2, \quad \text{with} \quad y(0) = 0,$$

is a case in point, amongst the most simple examples. In Exercise 2, the reader is asked to show that the method of proof of our main theorem provides a practical method of seeking a solution of this equation. In general, the theorem provides information about existence and uniqueness without the need for any attempt at integration whatsoever.

2.1 First-order differential equations in a single independent variable

We consider the existence and uniqueness of solutions $y = y(x)$ of the differential equation

(1) $$y' = f(x, y),$$

satisfying

(2) $$y(a) = c,$$

where a is a point in the domain of y and c is (another) constant. In order to achieve our aim, we must place restrictions on the function f:

(a) f is continuous in a region U of the (x, y)-plane which contains the rectangle

$$R = \{(x, y) : |x - a| \leq h,\ |y - c| \leq k\}$$

where h and k are positive constants,

(b) f satisfies the following 'Lipschitz condition' for all pairs of points $(x, y_1), (x, y_2)$ of U:

$$|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|,$$

where A is a (fixed) positive constant.

Restriction (a) implies that f must be bounded on R (by [A] of Chapter 0). Letting

$$M = \sup\{|f(x, y)| : (x, y) \in R\},$$

we add just one further restriction, to ensure, as we shall see, that the functions to be introduced are well defined:

(c) $$Mh \leq k.$$

We would also make a remark about the ubiquity of restriction (b). Such a Lipschitz condition must always occur when the partial derivative $\dfrac{\partial f}{\partial y}$ exists as a bounded function on U: if a bound on its modulus is $P > 0$, we can use the Mean Value Theorem of the differential calculus ([B] of Chapter 0), as applied to $f(x, y)$ considered as a function of y alone, to write, for some y_0 between y_1 and y_2,

$$|f(x, y_1) - f(x, y_2)| = \left|\frac{\partial f}{\partial y}(x, y_0)\right| |y_1 - y_2| \leq P|y_1 - y_2|.$$

Theorem 1 (Cauchy–Picard) When the restrictions (a), (b), (c) are applied, there exists, for $|x-a| \leq h$, a solution to the problem consisting of the differential equation (1) together with the boundary condition (2). The solution is unique amongst functions with graphs lying in U.

Proof We apply Picard's method (see section 1.2) and define a sequence (y_n) of functions $y_n : [a-h, a+h] \to \mathbb{R}$ by the iteration:

$$\begin{aligned} y_0(x) &= c, \\ y_n(x) &= c + \int_a^x f(t, y_{n-1}(t))\, dt, \qquad (n \geq 1). \end{aligned}$$

As f is continuous, $f(t, y_{n-1}(t))$ is a continuous function of t whenever $y_{n-1}(t)$ is. So, as in section 1.2, the iteration defines a sequence (y_n) of continuous functions on $[a-h, a+h]$, *provided that* $f(t, y_{n-1}(t))$ is defined on $[a-h, a+h]$; that is, provided that

$$|y_n(x) - c| \;\leq\; k, \quad \text{for all } x \in [a-h, a+h] \text{ and } n = 1, 2, \ldots$$

To see that this is true, we work by induction. Clearly,

$$|y_0(x) - c| \;\leq\; k, \quad \text{for each } x \in [a-h, a+h].$$

If $|y_{n-1}(x) - c| \leq k$ for all $x \in [a-h, a+h]$, where $n \geq 1$, then $f(t, y_{n-1}(t))$ is defined on $[a-h, a+h]$ and, for x in this interval,

$$|y_n(x) - c| \;=\; \left| \int_a^x f(t, y_{n-1}(t))\, dt \right| \;\leq\; M|x-a| \;\leq\; Mh \;\leq\; k, \quad (n \geq 1)$$

where we have used [D] of Chapter 0. The induction is complete.

What remains of the proof exactly parallels the procedure in section 1.2 and the reader is asked to fill in the details.

We provide next the inductive step of the proof of

$$|y_n(x) - y_{n-1}(x)| \;\leq\; \frac{A^{n-1}M}{n!}\, |x-a|^n, \quad \text{for all } x \in [a-h, a+h] \text{ and all } n \geq 1.$$

Suppose that

$$|y_{n-1}(x) - y_{n-2}(x)| \;\leq\; \frac{A^{n-2}M}{(n-1)!}\, |x-a|^{n-1} \quad \text{for all } x \in [a-h, a+h], \text{ where } n \geq 2.$$

Then, using the Lipschitz condition (b),

$$\begin{aligned} |y_n(x) - y_{n-1}(x)| &= \left| \int_a^x (f(t, y_{n-1}(t)) - f(t, y_{n-2}(t)))\, dt \right| \\ (3) \qquad &\leq \left| \int_a^x |f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))|\ dt \right| \\ &\leq \left| \int_a^x A\, |y_{n-1}(t) - y_{n-2}(t)|\ dt \right| \\ &\leq A \cdot \frac{A^{n-2}M}{(n-1)!} \left| \int_a^x |t-a|^{n-1}\ dt \right| \\ (4) \qquad &= \frac{A^{n-1}M}{n!}\, |x-a|^n, \qquad (n \geq 2) \end{aligned}$$

for every x in $[a-h, a+h]$.

Note The reader may wonder why we have kept the outer modulus signs in (3) above, after an application of [D] of Chapter 0. The reason is that it is possible for x to be less than a, while remaining in the interval $[a-h, a+h]$. Putting

$$S = f(t, y_{n-1}(t)) - f(t, y_{n-2}(t)), \quad (n \geq 2)$$

[D] is actually being applied as follows when $x < a$:

$$\left| \int_a^x S\, dt \right| = \left| -\int_x^a S\, dt \right| = \left| \int_x^a S\, dt \right| \leq \int_x^a |S|\ dt = -\int_a^x |S|\ dt = \left| \int_a^x |S|\ dt \right|.$$

Similarly, for $x < a$,

$$\left| \int_a^x |t-a|^{n-1}\ dt \right| = \left| -\int_x^a (a-t)^{n-1}\, dt \right| = \frac{(a-x)^n}{n} = \frac{|x-a|^n}{n},$$

establishing (4).

Continuing with the proof and putting, for each $n \geq 1$,

$$M_n \;=\; \frac{A^{n-1}Mh^n}{n!},$$

we see that we have shown that $|y_n(x) - y_{n-1}(x)| \leq M_n$ for all $n \geq 1$ and all x in $[a-h, a+h]$. However,

$$\sum_{n=1}^{\infty} M_n$$

is a series of constants, converging to

$$\frac{M}{A}\,(e^{Ah} - 1).$$

So, the Weierstrass M-test ([H] of Chapter 0) may again be applied to deduce that the series

$$\sum_{n=1}^{\infty}(y_n - y_{n-1}),$$

converges uniformly on $[a-h, a+h]$. Hence, as in section 1.2, the sequence (y_n) converges uniformly to, say, y on $[a-h, a+h]$. As each y_n is continuous (see above) so is y by [I](a) of Chapter 0. Further, $y_n(t)$ belongs to the closed interval $[c-k, c+k]$ for each n and each $t \in [a-h, a+h]$. Hence, $y(t) \in [c-k, c+k]$ for each $t \in [a-h, a+h]$, and $f(t, y(t))$ is a well-defined continuous function on $[a-h, a+h]$. Using the Lipschitz condition, we see that

$$|f(t, y(t)) - f(t, y_n(t))| \;\leq\; A|y(t) - y_n(t)| \qquad (n \geq 0)$$

for each t in $[a-h, a+h]$; so, the sequence $(f(t, y_n(t)))$ converges uniformly to $f(t, y(t))$ on $[a-h, a+h]$. Applying [I](a) of Chapter 0,

$$\int_a^x f(t, y_{n-1}(t))\,dt \to \int_a^x f(t, y(t))\,dt$$

as $n \to \infty$. So, letting $n \to \infty$ in the equation

$$y_n(x) \;=\; c + \int_a^x f(t, y_{n-1}(t))\,dt$$

defining our iteration, we obtain

$$y(x) \;=\; c + \int_a^x f(t, y(t))\,dt. \tag{5}$$

Note that $y(a) = c$. As the integrand in the right-hand side is continuous, we may (by [G] of Chapter 0) differentiate with respect to x to obtain

$$y'(x) = f(x, y(x)).$$

Thus $y = y(x)$ satisfies the differential equation (1) together with the condition (2). We have shown that there *exists* a solution to the problem.

The *uniqueness* of the solution again follows the pattern of our work in section 1.2. If $y = Y(x)$ is a second solution of (1) satisfying (2) with graph lying in U, then as $y(x)$ and $Y(x)$ are both continuous functions on the closed and bounded interval $[a-h, a+h]$, there must (by [A] of Chapter 0) be a constant N such that

$$|y(x) - Y(x)| \ \leq \ N \quad \text{for all } x \in [a-h, a+h].$$

Integrating $Y'(t) = f(t, Y(t))$ with respect to t, from a to x, we obtain

$$Y(x) = c + \int_a^x f(t, Y(t))\, dt,$$

since $Y(a) = c$. So, using (5) and the Lipschitz condition made available to us by the graph of $y = Y(x)$ lying in U,

$$\begin{aligned} |y(x) - Y(x)| \ &= \ \left| \int_a^x (f(t, y(t)) - f(t, Y(t)))\, dt \right| \\ &\leq \ \left| \int_a^x A\, |y(t) - Y(t)| \ dt \right| \qquad (6) \\ &\leq \ AN\, |x - a|\,, \quad \text{for all } t \in [a-h, a+h]. \end{aligned}$$

We leave it to the reader to show by induction that, for every integer n and every $x \in [a-h, a+h]$,

$$|y(x) - Y(x)| \leq \frac{A^n N}{n!}\, |x - a|^n\,.$$

As the right-hand side of this inequality may be made arbitrarily small, $y(x) = Y(x)$ for each x in $[a-h, a+h]$. Our solution is thus unique. □

Note (a) For continuous y, the differential equation (1) together with the condition (2) is equivalent to the integral equation (5).

(b) The analysis is simplified and condition (c) omitted if f is bounded and satisfies the Lipschitz condition in the strip

$$\{(x, y) : a - h \leq x \leq a + h\}.$$

(c) Notice that if the domain of f were sufficiently large and $\dfrac{\partial f}{\partial y}$ were to exist and be bounded there, then our work in the paragraph prior to the statement of the theorem would allow us to dispense with the graph condition for uniqueness.

Exercise 1 Find *all* the solutions of the differential equation

$$\frac{dy}{dx} \;=\; 3y^{2/3}$$

subject to the condition $y(0) = 0$. Which of the above restrictions does $f(x, y) = 3y^{2/3}$ not satisfy and why?

Exercise 2 By applying the method of proof of the above theorem, find the first three (non-zero) terms in the series expansion of the solution to the Riccati equation

$$y' \;=\; 1 + xy^2$$

satisfying $y(0) = 0$.

Exercise 3 Consider the initial value problem of finding a solution $y = y(x)$ in some neighbourhood of $x = 0$ to

$$\frac{dy}{dx} \;=\; f(x, y), \quad y(0) \;=\; c \quad (|x| < L)$$

where $f(x, y)$ is a continuous bounded real-valued function satisfying the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \;\leq\; A|y_1 - y_2| \quad (|x| < L,\ \text{all } y_1, y_2)$$

for some positive constant A. For each of the following special cases

(i) $f \;=\; xy, \qquad c = 1$

(ii) $f \;=\; xy^2, \qquad c = 1$

(iii) $f \;=\; xy^{\frac{1}{2}}, \qquad c = 0$

determine if the Lipschitz condition is satisfied, find all the solutions of the problem and specify the region of validity of each solution.

Exercise 4 Show that the problem

$$y' = f(y), \qquad y(0) = 0$$

has an infinite number of solutions $y = y(x)$, for $x \in [0, a]$, if

(i) $$f(y) = \sqrt{1+y} \quad \text{and} \quad a > 2,$$

or

(ii) $$f(y) = \sqrt{|y^2 - 1|} \quad \text{and} \quad a > \frac{\pi}{2}$$

[Note that, in case (ii), the function $y = y(x)$ given by

$$y(x) = \begin{cases} \sin x & (0 \le x < \frac{1}{2}\pi) \\ 1 & (\frac{1}{2}\pi \le x \le a) \end{cases}$$

is one solution of the problem.]

2.2 Two simultaneous equations in a single variable

It should be said at the outset that the methods of this section can be applied directly to the case of any finite number of simultaneous equations. The methods involve a straightforward extension of those employed in the last section and, for this reason, many of the details will be left for the reader to fill in.

We now seek solutions $y = y(x), z = z(x)$ of the simultaneous differential equations

(7) $$y' = f(x, y, z), \qquad z' = g(x, y, z)$$

which satisfy

(8) $$y(a) = c, \qquad z(a) = d,$$

where a is a point in the domains of y and z, c and d are also constants, and where

(d) f and g are continuous in a region V of (x, y, z)-space which contains the cuboid

$$S = \{(x, y, z) : |x - a| \le h,\ \max(|y - c|, |z - d|) \le k\}$$

where h, k are non-negative constants,

(e) f and g satisfy the following Lipschitz conditions at all points of V:

$$|f(x, y_1, z_1) - f(x, y_2, z_2)| \le A \max(|y_1 - y_2|, |z_1 - z_2|)$$
$$|g(x, y_1, z_1) - g(x, y_2, z_2)| \le B \max(|y_1 - y_2|, |z_1 - z_2|)$$

where A and B are positive constants,

(f) $$\max(M, N) \,.\, h \;\leq\; k,$$

where $M = \sup\{|f(x,y,z)| : (x,y,z) \in S\}$ and $N = \sup\{|g(x,y,z)| : (x,y,z) \in S\}$.

It is convenient (especially in the n-dimensional extension!) to employ the vector notation

$$\mathbf{y} \;=\; (y,z), \quad \mathbf{f} \;=\; (f,g), \quad \mathbf{c} \;=\; (c,d), \quad \mathbf{A} \;=\; (A,B), \quad \mathbf{M} \;=\; (M,N).$$

The reader can then easily check that, with use of the 'vector norm',

$$|\mathbf{y}| \;=\; \max(|y|,|z|),$$

where $\mathbf{y} = (y,z)$, the above problem reduces to

(7′) $$\mathbf{y}' \;=\; \mathbf{f}(x,\mathbf{y}),$$

satisfying

(8′) $$\mathbf{y}(a) \;=\; \mathbf{c}$$

where

(d′) $\mathbf{f}$ is continuous in a region V containing

$$S = \{(x,\mathbf{y}) : |x-a| \leq h,\ |\mathbf{y}-\mathbf{c}| \leq k\}$$

(e′) $\mathbf{f}$ satisfies the Lipschitz condition at all points of V:

$$|\mathbf{f}(x,\mathbf{y}_1) - \mathbf{f}(x,\mathbf{y}_2)| \leq |\mathbf{A}||\mathbf{y}_1 - \mathbf{y}_2|,$$

and

(f′) $$|\mathbf{M}|h \;\leq\; k.$$

The existence of a unique solution to (7′) subject to (8′) can now be demonstrated by employing the methods of section 2.1 to the iteration

(9) $$\begin{aligned} \mathbf{y}_0(x) \;&=\; \mathbf{c}, \\ \mathbf{y}_n(x) \;&=\; \mathbf{c} + \int_a^x \mathbf{f}(t,\mathbf{y}_{n-1}(t))\,dt, \qquad (n \geq 1) \end{aligned}$$

We thus have the following extension of Theorem 1.

Theorem 2 When the restrictions (d), (e), (f) are applied, there exists, for $|x-a| \leq h$, a solution to the problem consisting of the simultaneous differential equations (7) together with the boundary conditions (8). The solution is unique amongst functions with graphs lying in V.

Exercise 5 Consider the problem consisting of the simultaneous differential equations

$$y' = -yz, \quad z' = z^2$$

which satisfy

$$y(0) = 2, \quad z(0) = 3.$$

(i) Use Theorem 2 to prove that there is a unique solution to the problem on an interval containing 0.

(ii) Find the solution to the problem, specifying where this solution exists.

Exercise 6 Consider the problem

$$y' = 2 - yz, \quad z' = y^2 - xz, \quad y(0) = -1, \quad z(0) = 2.$$

Find the first three iterates $\mathbf{y}_0(x)$, $\mathbf{y}_1(x)$, $\mathbf{y}_2(x)$ in the vector iteration (9) corresponding to this problem.

Exercise 7 With the text's notation, prove that, if $f = f(x, y, z) = f(x, \mathbf{y})$ has continuous bounded partial derivatives in V, then f satisfies a Lipschitz condition on S of the form given in (e′).
[HINT: Write $f(x, y_1, z_1) - f(x, y_2, z_2) = f(x, y_1, z_1) - f(x, y_2, z_1) + f(x, y_2, z_1) - f(x, y_2, z_2)$ and use the Mean Value Theorem of the differential calculus ([B] of Chapter 0).]

Exercise 8 Compare the method employed in the text for solving (7) subject to (8) with that given by the simultaneous iterations

$$y_0(x) = c,$$

$$y_n(x) = c + \int_a^x f(t, y_{n-1}(t), z_{n-1}(t))\, dt,$$

and

$$z_0(x) = d,$$

$$z_n(x) = d + \int_a^x g(t, y_{n-1}(t), z_{n-1}(t))\, dt,$$

for $n \geq 1$ and $x \in [a-h, a+h]$. In particular, find bounds for $|y_1 - y_0|$, $|y_2 - y_1|$ and $|y_3 - y_2|$ on $[a-h, a+h]$ in terms of A, B, M, N and h.

2.3 A second-order equation

We now use Theorem 2 to find a solution $y = y(x)$ to the problem consisting of the differential equation

$$\frac{d^2y}{dx^2} \equiv y'' = g(x, y, y') \tag{10}$$

together with the initial conditions

$$y(a) = c, \quad y'(a) = d, \quad (c, d \text{ constants}). \tag{11}$$

(Note that y and y' are both given at the same point a.)

A problem of the type given by (10) taken together with 'initial conditions' (11), when a solution is only required for $x \geq a$, is called an **initial value problem** (IVP) – the variable x can be thought of as time (and customarily is then re-named t).

The trick is to convert (10) to the pair of simultaneous equations

$$y' = z, \quad z' = g(x, y, z), \tag{10$'$}$$

(the first equation defining the new function z). Corresponding to (11) we have

$$y(a) = c, \quad z(a) = d. \tag{11$'$}$$

We, of course, require certain restrictions on $g = g(x, y, z)$:

(d$''$) g is continuous on a region V of (x, y, z)-space which contains

$$S = \{(x, y, z) : |x - a| \leq h, \ \max(|y - c|, |z - d|) \leq k\}$$

where h, k are non-negative constants,

(e$''$) g satisfies the following Lipschitz condition at all points of V:

$$|g(x, y_1, z_1) - g(x, y_2, z_2)| \leq B \max(|y_1 - y_2|, |z_1 - z_2|),$$

where B is a constant.

Theorem 3 When the restrictions (d$''$), (e$''$) are imposed, then, for some $h > 0$, there exists, when $|x - a| \leq h$, a solution to the problem consisting of the second-order differential equation (10) together with the initial conditions (11). The solution is unique amongst functions with graphs lying in V.

The reader should deduce Theorem 3 from Theorem 2. It will be necessary, in particular, to check that $f(x,y,z) = z$ satisfies a Lipschitz condition on V and that an h can be found so that (f) of section 2.2 can be satisfied. We should note that the methods of this section can be extended so as to apply to the nth-order equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),$$

when subject to initial conditions

$$y(a) = c_0, \quad y'(a) = c_1, \ \ldots, \quad y^{(n-1)}(a) = c_{n-1}.$$

We conclude our present discussion of the second-order equation by considering the special case of the *non-homogeneous linear equation*

$$(12) \qquad p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x), \qquad (x \in [a,b])$$

where p_0, p_1, p_2, f are continuous on $[a,b]$, $p_2(x) > 0$ for each x in $[a,b]$, and the equation is subject to the initial conditions

$$(13) \qquad y(x_0) = c, \quad y'(x_0) = d, \qquad (x_0 \in [a,b]; \quad c, d \text{ constants}).$$

Theorem 4 There exists a unique solution to the problem consisting of the linear equation (12) together with the initial conditions (13).

As continuity of the function

$$g(x,y,z) = \frac{f(x)}{p_2(x)} - \frac{p_0(x)}{p_2(x)}\, y - \frac{p_1(x)}{p_2(x)}\, z$$

is clear, the reader need only check that this same function satisfies the relevant Lipschitz condition for all x in $[a,b]$. Note that the various continuous functions of x are bounded on $[a,b]$. (No such condition as (f) of section 2.2 is here necessary, nor is the graph condition of Theorem 3.)

By and large, the differential equations that appear in these notes are linear. It is of special note that the unique solution obtained for the equation of Theorem 4 is valid for the whole interval of definition of that linear equation. In our other theorems, although existence is only given 'locally' (for example, where $|x - a| \leq h$ *and* $Mh \leq k$ in Theorem 1), solutions are often valid in a larger domain. Often the argument used above to establish uniqueness in a limited domain can be used to extend this uniqueness to wherever a solution exists.

Exercise 9 Consider the problem

$$yy'' = -(y')^2, \quad y(0) = y'(0) = 1.$$

(i) Use Theorem 3 to show that the problem has a unique solution on an interval containing 0.

(ii) Find the solution and state where it exists.

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3 The Homogeneous Linear Equation and Wronskians

The main aim of this chapter will be to use Theorem 4 of Chapter 2 – and specifically both existence and uniqueness of solutions – to develop a theory that will describe the solutions of the **homogeneous linear second-order equation**

$$p_2(x)y'' + p_1(x)y' + p_0(x)y \;=\; 0, \qquad (x \in [a,b]) \tag{1}$$

where p_0, p_1, p_2 are continuous real-valued functions on $[a,b]$ and $p_2(x) > 0$ for each x in $[a,b]$. ('Homogeneous' here reflects the zero on the right-hand side of the equation which allows λy to be a solution (for any real constant λ) whenever y is a given solution.) The language of elementary linear algebra will be used and the theory of simultaneous linear equations will be presumed.

Central to our discussion will be the Wronskian, or Wronskian determinant: if $y_1 : [a,b] \to \mathbb{R}$ and $y_2 : [a,b] \to \mathbb{R}$ are differentiable functions on the closed interval $[a,b]$, the *Wronskian of* y_1 *and* y_2, $W(y_1, y_2) : [a,b] \to \mathbb{R}$, is defined, for $x \in [a,b]$, by

$$W(y_1,y_2)(x) \;=\; \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \;=\; y_1(x)y_2'(x) - y_1'(x)y_2(x). \tag{2}$$

If y_1 and y_2 are solutions of (1), it will turn out that *either* $W(y_1, y_2)$ is identically zero *or* never zero in $[a,b]$.

3.1 Some linear algebra

Suppose that $y_i : [a, b] \to \mathbb{R}$ and that c_i is a real constant, for $i = 1, \ldots, n$. By

$$c_1 y_1 + \ldots + c_n y_n \;=\; 0$$

is meant

$$c_1 y_1(x) + \ldots + c_n y_n(x) \;=\; 0 \quad \text{for each } x \text{ in } [a, b].$$

We may describe this situation by saying that

$$c_1 y_1 + \ldots + c_n y_n \quad \text{or} \quad \sum_{i=1}^{n} c_i y_i \;\; \textit{is identically zero on } [a, b].$$

If $y_i : [a, b] \to \mathbb{R}$ for $i = 1, \ldots, n$, the set $\{y_1, \ldots, y_n\}$ is *linearly dependent on* $[a, b]$ if and only if there are real constants $c_1, \ldots, c_n$, not all zero, such that

$$c_1 y_1 + \ldots + c_n y_n \;=\; 0.$$

Otherwise, the set is *linearly independent on* $[a, b]$. So, $\{y_1, \ldots, y_n\}$ is linearly independent on $[a, b]$ if and only if

$$c_1 y_1 + \ldots + c_n y_n \;=\; 0,$$

with $c_1, \ldots, c_n$ real constants, necessitates

$$c_1 \;=\; \ldots \;=\; c_n \;=\; 0.$$

It is a common abuse of language to say that $y_1, \ldots, y_n$ *are* linearly dependent (or independent) when one means that the *set* $\{y_1, \ldots, y_n\}$ *is* linearly dependent (independent). We shall find ourselves abusing language in this manner.

We now turn to stating some elementary results relating to solutions $\{x_1, \ldots, x_n\}$ of the system of simultaneous linear equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & \ldots & + & a_{1n}x_n & = & 0 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \ldots & + & a_{mn}x_n & = & 0 \end{array}$$

where a_{ij} is a real constant for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

(a) When $m = n$, the system has a solution *other than* the 'zero solution'

$$x_1 = \ldots = x_n = 0$$

if and only if the 'determinant of the coefficients' is zero, that is,

$$\begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} = 0.$$

(b) When $m < n$, the system has a solution other than the zero solution.

We conclude this section with an application of result (a), which will give us our first connection between linear dependence and Wronskians.

Proposition 1 If y_1, y_2 are differentiable real-valued functions, linearly dependent on $[a, b]$, then $W(y_1, y_2)$ is identically zero on $[a, b]$.

Proof As y_1, y_2 are linearly dependent, there are real constants c_1, c_2, not both zero, such that

(3) $$c_1 y_1(x) + c_2 y_2(x) = 0, \quad \text{for each } x \text{ in } [a, b].$$

Differentiating with respect to x we have

(4) $$c_1 y_1'(x) + c_2 y_2'(x) = 0, \quad \text{for each } x \text{ in } [a, b].$$

Treat the system consisting of (3) and (4) as equations in c_1 and c_2 (in the above notation, take $x_1 = c_1$, $x_2 = c_2$, $a_{11} = y_1(x)$, $a_{12} = y_2(x)$, $a_{21} = y_1'(x)$, $a_{22} = y_2'(x)$). Since c_1 and c_2 are not both zero, we may use result (a) to deduce that the determinant of the coefficients of c_1 and c_2 is zero. However, this determinant is precisely $W(y_1, y_2)(x)$, the Wronskian evaluated at x. We have thus shown that $W(y_1, y_2)(x) = 0$, for each x in $[a, b]$; that is, that $W(y_1, y_2)$ is identically zero on $[a, b]$. □

The converse of Proposition 1 does not hold: the reader is asked to demonstrate this by providing a solution to the second exercise below.

Exercise 1 If $y_1(x) = \cos x$ and $y_2(x) = \sin x$ for $x \in [0, \pi/2]$, show that $\{y_1, y_2\}$ is linearly independent on $[0, \pi/2]$.

Exercise 2 Define $y_1 : [-1,1] \to \mathbb{R}$, $y_2 : [-1,1] \to \mathbb{R}$ by

$$\begin{aligned} y_1(x) &= x^3, & y_2(x) &= 0 & &\text{for } x \in [0,1], \\ y_1(x) &= 0, & y_2(x) &= x^3 & &\text{for } x \in [-1,0]. \end{aligned}$$

Show that y_1 and y_2 are twice continuously differentiable functions on $[-1,1]$, that $W(y_1,y_2)$ is identically zero on $[-1,1]$, but that $\{y_1,y_2\}$ is linearly independent on $[-1,1]$.

3.2 Wronskians and the linear independence of solutions of the second-order homogeneous linear equation

We commence this section with an elementary result which is useful in any discussion of homogeneous linear equations. The proof is left to the reader.

Proposition 2 If $y_1, \ldots, y_n$ are solutions of (1) and $c_1, \ldots, c_n$ are real constants, then

$$c_1 y_1 + \ldots + c_n y_n$$

is also a solution of (1).

We now show that the converse of Proposition 1 holds when y_1 and y_2 are solutions of (1). The result we prove looks at first sight (and misleadingly) stronger.

Proposition 3 If y_1, y_2 are solutions of the linear equation (1) and if $W(y_1,y_2)(x_0) = 0$ for some x_0 in $[a,b]$, then y_1 and y_2 are linearly dependent on $[a,b]$ (and hence $W(y_1,y_2)$ is identically zero on $[a,b]$).

Proof Consider the following system as a pair of equations in c_1 and c_2:

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 0, \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= 0. \end{aligned} \tag{5}$$

(Note that $y_1'(x_0), y_2'(x_0)$ exist, as y_1, y_2 both solve (1) and hence even have second derivatives.) The determinant of the coefficients, here $W(y_1,y_2)(x_0)$, is zero. So, using result (a) of section 3.1, there is a solution

$$c_1 = C_1, \quad c_2 = C_2 \tag{6}$$

with C_1, C_2 not both zero.

Proposition 2 allows us to conclude that the function $y : [a, b] \to \mathbb{R}$ defined by

$$y(x) \;=\; C_1 y_1(x) + C_2 y_2(x), \qquad (x \in [a, b])$$

is a solution of (1). But notice that equations (5), taken with the solution (6), state that

$$y(x_0) \;=\; y'(x_0) \;=\; 0. \tag{7}$$

One solution of (1) together with the initial conditions (7) is clearly $y = 0$ identically on $[a, b]$. By Theorem 4 of Chapter 2, there can be no others; and so, necessarily,

$$C_1 y_1 + C_2 y_2 \;=\; 0$$

identically on $[a, b]$. Recalling that not both of C_1 and C_2 are zero, we have shown that y_1 and y_2 are linearly dependent on $[a, b]$. □

Propositions 1 and 3 are so important that we re-state them together as the following proposition.

Proposition 4 Suppose that y_1 and y_2 are solutions of the linear equation (1). Then
(a) $W(y_1, y_2)$ is identically zero on $[a, b]$ or never zero on $[a, b]$;
(b) $W(y_1, y_2)$ is identically zero on $[a, b]$ if and only if y_1 and y_2 are linearly dependent on $[a, b]$.

A proof similar to that of Proposition 3 establishes the following result which shows that we can never have more than two solutions to (1) which are linearly independent on $[a, b]$.

Proposition 5 If $n > 2$ and $y_1, \ldots, y_n$ are solutions of (1), then $\{y_1, \ldots, y_n\}$ is a linearly dependent set on $[a, b]$.

Proof Pick x_0 in $[a, b]$ and consider the pair of equations

$$c_1 y_1(x_0) + \ldots + c_n y_n(x_0) \;=\; 0$$

$$c_1 y_1'(x_0) + \ldots + c_n y_n'(x_0) \;=\; 0$$

in $c_1, \ldots, c_n$. As $n > 2$, result (b) of the last section implies that there is a solution

$$c_1 \;=\; C_1, \;\ldots,\; c_n \;=\; C_n$$

with $C_1, \ldots, C_n$ not all zero.

Using Proposition 2 above and Theorem 4 of Chapter 2, we deduce, as in the proof of Proposition 3, that

$$C_1 y_1 + \ldots + C_n y_n \;=\; 0$$

identically on $[a, b]$ and hence that $y_1, ..., y_n$ are linearly dependent on $[a, b]$. □

We conclude our discussion of the solution of (1) by using Wronskians to show that this linear equation actually possesses two linearly independent solutions.

Proposition 6 There exist two solutions y_1, y_2 of (1) which are linearly independent on $[a, b]$. Further, any solution y of (1) may be written in terms of y_1 and y_2 in the form

$$y \;=\; c_1 y_1 + c_2 y_2, \tag{8}$$

where c_1 and c_2 are constants.

Proof Pick x_0 in $[a, b]$. The existence part of Theorem 4 of Chapter 2 produces a solution y_1 of (1) satisfying the initial conditions

$$y_1(x_0) \;=\; 1, \qquad y_1'(x_0) \;=\; 0.$$

Similarly, there is a solution y_2 of (1) satisfying

$$y_2(x_0) \;=\; 0, \qquad y_2'(x_0) \;=\; 1.$$

As $W(y_1, y_2)(x_0) = 1$, we may use Proposition 1 to deduce that y_1 and y_2 are linearly independent on $[a, b]$.

If y is any solution of (1) then $\{y, y_1, y_2\}$ is linearly dependent on $[a, b]$ by Proposition 5. So, there are constants c, c_1', c_2' not all zero, such that

$$cy + c_1' y_1 + c_2' y_2 \;=\; 0.$$

The constant c must be non-zero; for otherwise,

$$c_1' y_1 + c_2' y_2 \;=\; 0,$$

which would necessitate, as $\{y_1, y_2\}$ is linearly independent, $c_1' = c_2' = 0$, contradicting the fact that not all of c, c_1', c_2' are zero. We may therefore define $c_1 = -c_1'/c$ and $c_2 = -c_2'/c$ to give

$$y \;=\; c_1 y_1 + c_2 y_2.$$ □

Note There are other ways of proving the propositions of this chapter. The proofs here have been chosen as they can be extended directly to cover the case of the n-th order homogeneous linear equation.

Exercise 3 Find the Wronskian $W(y_1, y_2)$ corresponding to linearly independent solutions y_1, y_2 of the following differential equations in $y = y(x)$, satisfying the given conditions. Methods for solving the equations may be found in the Appendix.

(a) $y'' = 0, \qquad y_1(-1) = y_2(1) = 0$

(b) $y'' - y = 0, \qquad y_1(0) = y_2'(0) = 0$

(c) $y'' + 2y' + (1 + k^2)y = 0, \qquad y_1(0) = y_2(\pi) = 0, \qquad$ for $k > 0$,

(d) $x^2y'' + xy' - k^2y = 0, \qquad y_1(1) = y_2(2) = 0, \qquad$ for $x, k > 0$.

(Note that the value of the Wronskian may still depend on constants of integration.)

Exercise 4 By considering (separately) the differential equations

$$y'' + y^2 = 0, \qquad y'' = 1,$$

show that linearity and homogeneity of (1) are necessary hypotheses in Proposition 2.

Exercise 5 Suppose that y_1, y_2 are solutions of (1). Show that the Wronskian $W = W(y_1, y_2)$ is a solution of the differential equation

$$p_2(x)W' + p_1(x)W = 0, \qquad (x \in [a, b]). \tag{9}$$

Exercise 6 By first solving (9) of Exercise 5, give an alternative proof of Proposition 4(a).

Exercise 7 Show that at least one of the components of one of the solutions (y, z) of the simultaneous equations

$$\frac{dz}{dx} = \alpha \frac{df}{dx} \cdot \frac{dy}{dx} - xy, \qquad f(x)\frac{dy}{dx} = z,$$

where $f(x)$ is continuous, everywhere positive, and unbounded as $x \to \infty$, is unbounded as $x \to \infty$ if $\alpha > 0$.

[HINT: Form a second order equation for y and then solve the corresponding equation (9) for the Wronskian of its solutions.]

Exercise 8 Describe how equation (9) of Exercise 5 may be used to find the general solution of (1) when one (nowhere zero) solution $y = u$ is known. Compare this method with the one given in section (5) of the Appendix by showing that, if $y = u$ and $y = uv$ are solutions of (1), then (9) for $W = W(u, uv)$ gives rise to

$$p_2u^2v'' + (2uu'p_2 + p_1u^2)v' \;=\; 0.$$

4 The Non-Homogeneous Linear Equation

In this chapter we shall consider the *non-homogeneous second-order linear equation*

$$(1) \qquad p_2(x)y'' + p_1(x)y' + p_0(x)y \;=\; f(x), \qquad (x \in [a,b])$$

where $f : [a,b] \to \mathbb{R}$ and each $p_i : [a,b] \to \mathbb{R}$ are continuous, and $p_2(x) > 0$, for each x in $[a,b]$. Any particular solution of (1) (or, indeed, of any differential equation) is called a *particular integral* of the equation, whereas the general solution $c_1y_1 + c_2y_2$ of the corresponding homogeneous equation

$$(2) \qquad p_2(x)y'' + p_1(x)y' + p_0(x)y \;=\; 0, \qquad (x \in [a,b])$$

given by Proposition 6 of Chapter 3 (where y_1, y_2 are linearly independent solutions of (2) and c_1, c_2 are arbitrary real constants) is called the *complementary function* of (1). If y_P is any particular integral and y_C denotes the complementary function of (1), $y_C + y_P$ is called the *general solution* of (1). We justify this terminology by the following proposition, which shows that, once two linearly independent solutions of (2) are found, all that remains to be done in solving (1) is to find one particular integral.

Proposition 1 Suppose that y_P is any particular integral of the non-homogeneous linear equation (1) and that y_1, y_2 are linearly independent solutions of the corresponding homogeneous equation (2). Then

(a) $c_1y_1 + c_2y_2 + y_P$ is a solution of (1) for any choice of the constants $c_1,\ c_2$,

(b) if y is any solution (that is, any particular integral) of (1), there exist (particular) real constants $C_1,\ C_2$ such that

$$y \;=\; C_1y_1 + C_2y_2 + y_P.$$

Proof (a) As $c_1y_1 + c_2y_2$ is a solution of (2) for any real choice of c_1, c_2 by Proposition 2 of Chapter 3, all we need to show is that, if Y is a solution of (2) and y_P a solution of (1), then $Y + y_P$ is a solution of (1). In these circumstances, we have

$$p_2Y'' + p_1Y' + p_0Y \;=\; 0$$

and

(3) $$p_2y_P'' + p_1y_P' + p_0y_P \;=\; f.$$

By adding,

$$p_2(Y+y_P)'' + p_1(Y+y_P)' + p_0(Y+y_P) \;=\; f;$$

and thus, $Y + y_P$ is a solution of (1) as required.

(b) As well as (3) above, we are given that

(4) $$p_2y'' + p_1y' + p_0y \;=\; f.$$

Subtracting,

$$p_2(y-y_P)'' + p_1(y-y_P)' + p_0(y-y_P) \;=\; 0;$$

so that, $y - y_P$ solves (2). Proposition 6 of Chapter 3 then finds real constants $C_1,\ C_2$ such that

$$y - y_P \;=\; C_1y_1 + C_2y_2.$$

The proposition is established. □

4.1 The method of variation of parameters

The reader will probably already have encountered methods for finding particular integrals for certain (benign!) functions $f(x)$ occurring on the right-hand side of equation (1). This section produces a systematic method for finding particular integrals once one has determined two linearly independent solutions of (2), that is, once one has already determined the complementary function of (1).

All there really is to the method is to note that, if y_1, y_2 are linearly independent solutions of (2), then a particular integral of (1) is $k_1 y_1 + k_2 y_2$, where $k_i : [a,b] \to \mathbb{R}$ $(i = 1, 2)$ are the continuously differentiable functions on $[a,b]$ defined, for any real constants α, β in $[a,b]$, by

$$(5) \qquad k_1(x) \;=\; -\int_\alpha^x \frac{y_2(t)f(t)}{p_2(t)W(t)}\,dt, \quad k_2(x) \;=\; \int_\beta^x \frac{y_1(t)f(t)}{p_2(t)W(t)}\,dt, \quad (x \in [a,b])$$

where

$$W \;=\; W(y_1, y_2) \;\equiv\; y_1 y_2' - y_2 y_1'$$

is the Wronskian of y_1 and y_2. (Notice that $W(t) \neq 0$ for each t in $[a,b]$ because y_1, y_2 are linearly independent.) To see that this is true, we differentiate k_1 and k_2 as given by (5):

$$k_1'(x) \;=\; -\frac{y_2(x)f(x)}{p_2(x)W(x)}, \qquad k_2'(x) \;=\; \frac{y_1(x)f(x)}{p_2(x)W(x)}, \qquad (x \in [a,b])$$

and notice that k_1', k_2' must therefore satisfy the simultaneous linear equations

$$(6) \qquad \begin{aligned} k_1'(x)y_1(x) + k_2'(x)y_2(x) &= 0, \\ k_1'(x)y_1'(x) + k_2'(x)y_2'(x) &= f(x)/p_2(x). \end{aligned} \qquad (x \in [a,b])$$

(The reader should check this.) Hence, putting

$$y \;=\; k_1 y_1 + k_2 y_2,$$

we have

$$y' \;=\; k_1 y_1' + k_2 y_2'$$

and

$$y'' \;=\; k_1 y_1'' + k_2 y_2'' + f/p_2.$$

So,

$$p_2y'' + p_1y' + p_0y \;=\; k_1(p_2y_1'' + p_1y_1' + p_0y_1) + k_2(p_2y_2'' + p_1y_2' + p_0y_2) + f \;=\; f,$$

since $y_1,\ y_2$ solve (2). Thus, $y = k_1y_1 + k_2y_2$ solves (1) and the general solution of (1) is

$$(7) \qquad y(x) \;=\; c_1y_1(x) + c_2y_2(x) - \int_\alpha^x \frac{y_1(x)y_2(t)}{p_2(t)W(t)}\,f(t)\,dt + \int_\beta^x \frac{y_1(t)y_2(x)}{p_2(t)W(t)}\,f(t)\,dt$$

where c_1 and c_2 are real constants and $x \in [a, b]$.

The constants α, β should not be regarded as arbitrary in the sense that c_1, c_2 are. Rather, they should be considered as part of the definition of k_1 and k_2. Notice that, if $y = y(x)$ is the particular integral given by formula (7) with $c_1 = C_1$ and $c_2 = C_2$, and if $\alpha',\ \beta'$ are in $[a, b]$ then

$$y(x) \;=\; C_1'y_1(x) + C_2'y_2(x) - \int_{\alpha'}^x \frac{y_1(x)y_2(t)}{p_2(t)W(t)}\,f(t)\,dt + \int_{\beta'}^x \frac{y_1(t)y_2(x)}{p_2(t)W(t)}\,f(t)\,dt$$

for each x in $[a, b]$, where C_1', C_2' are the *constants* given by

$$(8) \qquad C_1' \;=\; C_1 - \int_\alpha^{\alpha'} \frac{y_2(t)f(t)}{p_2(t)W(t)}\,dt, \qquad C_2' \;=\; C_2 + \int_\beta^{\beta'} \frac{y_1(t)f(t)}{p_2(t)W(t)}\,dt.$$

Thus, changes in $\alpha,\ \beta$ just make corresponding changes in $C_1,\ C_2$.

An appropriate choice of $\alpha,\ \beta$ can often depend on conditions applied to the solution of (1).

(a) **Initial conditions** Suppose we are given values for $y(a)$ and $y'(a)$. Then it can be most convenient to choose $\alpha = \beta = a$. The particular integral in (7) is then

$$(9) \qquad \int_a^x \frac{(y_1(t)y_2(x) - y_1(x)y_2(t))}{p_2(t)W(t)}\,f(t)\,dt \;=\; \int_a^x \frac{(y_1(t)y_2(x) - y_1(x)y_2(t))f(t)}{(y_1(t)y_2'(t) - y_2(t)y_1'(t))p_2(t)}\,dt.$$

The reader should check that this particular integral and its derivative with respect to x are both zero at $x = a$. This is technically useful when given the above initial conditions, making it easier to calculate the constants $c_1,\ c_2$ in this case.

(b) **Boundary conditions** Suppose now that we are given values for $y(a)$ and $y(b)$. Convenient choices for $\alpha,\ \beta$ are often $\alpha = b,\ \beta = a$. The particular integral then becomes

$$\text{(10)} \qquad \int_x^b \frac{y_1(x)y_2(t)}{p_2(t)W(t)}\, f(t)\, dt + \int_a^x \frac{y_1(t)y_2(x)}{p_2(t)W(t)}\, f(t)\, dt \;=\; \int_a^b G(x,t)f(t)\, dt,$$

where

$$G(x,t) \;=\; \begin{cases} \dfrac{y_1(x)y_2(t)}{p_2(t)W(t)}\,, & \text{for } a \le x \le t \le b, \\[2ex] \dfrac{y_1(t)y_2(x)}{p_2(t)W(t)}\,, & \text{for } a \le t \le x \le b. \end{cases}$$

Notice that, at $x = a$, the first integral on the left-hand side of (10) is a multiple of $y_1(a)$ and the second integral vanishes. At $x = b$, it is the first integral that vanishes and the second is a multiple of $y_2(b)$. This can be especially useful if the linearly independent functions y_1, y_2 can be chosen so that $y_1(a) = y(a)$ and $y_2(b) = y(b)$. We shall return to these matters and the function $G = G(x,t)$ in our discussion of Green's functions in the next section.

An alternative view of the method

A commonly found presentation of the method of variation of parameters is the following.

Suppose that y_1 and y_2 are linearly independent solutions of (2). As $y = c_1y_1 + c_2y_2$ is a solution (the general solution) of the homogeneous equation (2), 'it is natural' to seek a solution to the non-homogeneous equation (1) by 'varying the parameters' c_1 and c_2. So, we seek functions $k_1,\ k_2$ such that $y = k_1y_1 + k_2y_2$ is a particular solution of (1) (where k_1 and k_2 are continuously differentiable). Then

$$\text{(11)} \qquad y' \;=\; k_1y_1' + k_2y_2' + k_1'y_1 + k_2'y_2.$$

The next step in the argument is to stipulate that k_1 and k_2 are to be chosen in order that

$$\text{(12)} \qquad k_1'y_1 + k_2'y_2 \;=\; 0.$$

(Yes, this is possible! We are often told that there is sufficient freedom in our choice of k_1 and k_2 to allow this and hence, of course, to simplify the subsequent 'working'.) Differentiating again and using (12),

$$\text{(13)} \qquad y'' \;=\; k_1y_1'' + k_2y_2'' + k_1'y_1' + k_2'y_2'.$$

In order to derive a further condition to permit $y = k_1y_1+k_2y_2$ to be a solution to (1), we must substitute for this y, for y' given by (11) subject to (12), and for y'' given by (13) in equation (1). The reader should check that, since y_1 and y_2 solve (2), it is necessary that

$$k_1'y_1' + k_2'y_2' \;=\; f/p_2. \tag{14}$$

Equations (12) and (14) are of course just equations (6). These may be solved for k_1' and k_2' as the determinant of the coefficients is the Wronskian $W(y_1, y_2)$ of the linearly independent solutions $y_1,\ y_2$ and hence is non-zero everywhere in $[a, b]$. Reversing steps in our earlier discussion of the method,

$$k_1(x) - k_1(\alpha) \;=\; -\int_\alpha^x \frac{y_2(t)f(t)}{p_2(t)W(t)}\,dt, \qquad k_2(x) - k_2(\beta) \;=\; \int_\beta^x \frac{y_1(t)f(t)}{p_2(t)W(t)}\,dt$$

for each x in $[a, b]$. Thus, the general solution of (1) is given by

$$\begin{aligned} y(x) \;&=\; c_1y_1(x) + c_2y_2(x) + k_1(x)y_1(x) + k_2(x)y_2(x) \\ &=\; c_1'y_1(x) + c_2'y_2(x) - \int_\alpha^x \frac{y_1(x)y_2(t)}{p_2(t)W(t)}\,f(t)\,dt + \int_\beta^x \frac{y_1(t)y_2(x)}{p_2(t)W(t)}\,f(t)\,dt \end{aligned}$$

where $c_1' = c_1 + k_1(\alpha)$ and $c_2' = c_2 + k_2(\beta)$ are constants and $x \in [a, b]$.

This alternative view of the variation of parameters method does not present a purely deductive argument (in particular, it is necessary to stipulate (12)), but the reader may care to use it to aid his recall of k_1 and k_2 as defined in (5). The condition (12) will then need to be remembered!

It is our experience that computational errors in applying the method to practical problems are more easily avoided by deducing the general formula (7) before introducing particular functions $y_1,\ y_2$ and calculating $W(y_1, y_2)$.

It is also worth recalling that the variation of parameters particular integral is only one of many possible particular integrals. Practically, one should see if a particular integral can be more easily found otherwise. The Green's function method described in the next section, when applicable, is remarkably efficient.

Exercise 1 Check the following details:

(a) that equations (6) are satisfied by $k_1'(x)$ and $k_2'(x)$,

(b) that the values given for C_1' and C_2' in (8) are correct,

(c) that the integral (9) and its derivative with respect to x are both zero at $x = a$.

Use the method of variation of parameters to solve the following three problems.

Exercise 2 Find the general solution of $y'' + y = \tan x, \quad 0 < x < \pi/2.$

Exercise 3 Show that the solution of the equation

$$y'' + 2y' + 2y = f(x),$$

with f continuous and initial conditions $y(0) = y'(0) = 1$, can be written in the form

$$y(x) = e^{-x}(\cos x + 2\sin x) + \int_0^x e^{-(x-t)} \sin(x-t) f(t)\, dt.$$

Exercise 4 Find the necessary condition on the continuous function g for there to exist a solution of the equation

$$y'' + y = g(x),$$

satisfying $y(0) = y(\pi) = 0$.

Exercise 5 The function $y = y(x)$ satisfies the homogeneous differential equation

$$y'' + (1 - h(x))y = 0, \qquad (0 \leq x \leq K)$$

where h is a continuous function and K is a positive constant, together with the initial conditions $y(0) = 0,\ y'(0) = 1$. Using the variation of parameters method, show that y also satisfies the integral equation

$$y(x) = \sin x + \int_0^x y(t)h(t)\sin(x-t)\, dt \qquad (0 \leq x \leq K).$$

If $|h(x)| \leq H$ for $0 \leq x \leq K$ and some positive constant H, show that

$$|y(x)| \leq e^{Hx} \qquad (0 \leq x \leq K).$$

[HINTS: Re-write the differential equation in the form

$$y'' + y = h(x)y.$$

For the last part, use Picard's iterative method described in Chapter 1.]

4.2 Green's functions

Up to this point, in order to find the solution of a problem consisting of a differential equation together with initial or boundary conditions, we have first found a general solution of the equation and later fitted the other conditions (by finding values for constants occurring in the general solution in order that the conditions be met). The method produced by the theorem of this section builds the boundary condition into finding the solution from the start.

Theorem 1 Suppose that the operator L is defined by

$$Ly \;\equiv\; \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y \;\equiv\; (p(x)y')' + q(x)y,$$

where y is twice continuously differentiable, p is continuously differentiable, q is continuous and $p(x) > 0$ for all x in $[a,b]$. Suppose further that the homogeneous equation

$$\text{(H)} \qquad Ly \;=\; 0 \qquad (x \in [a,b])$$

has only the trivial solution (that is, the solution $y = 0$ identically on $[a,b]$) when subject to both boundary conditions

$$(\alpha) \qquad A_1 y(a) + B_1 y'(a) \;=\; 0$$

$$(\beta) \qquad A_2 y(b) + B_2 y'(b) \;=\; 0$$

where $A_1,\ A_2,\ B_1,\ B_2$ are constants ($A_1,\ B_1$ not both zero and $A_2,\ B_2$ not both zero). If f is continuous on $[a,b]$, then the non-homogeneous equation

$$\text{(N)} \qquad Ly \;=\; f(x), \qquad (x \in [a,b])$$

taken together with both conditions (α) and (β), has a *unique* solution which may be written in the form

$$y(x) \;=\; \int_a^b G(x,t)f(t)\,dt, \qquad (x \in [a,b])$$

where G is continuous on $[a,b]^2$, is twice continuously differentiable on $\{(x,y) \in [a,b]^2 : x \neq y\}$, and satisfies

$$\frac{\partial G}{\partial x}(t+0,t) - \frac{\partial G}{\partial x}(t-0,t) \;=\; \frac{1}{p(t)}, \qquad (t \in [a,b]).$$

[We have used the notation: when $f = f(x,t)$,

$$f(t+0,t) \;=\; \lim_{x\downarrow t} f(x,t), \qquad f(t-0,t) \;=\; \lim_{x\uparrow t} f(x,t).]$$

The linear equations (H) and (N) here, with their left-hand sides written in the form $(py')' + qy$, are said to be in *self-adjoint form*.

Note The problem consisting of trying to find solutions of (N) together with (α) and (β), conditions applying at the two distinct points a and b, is called a **two-point boundary value problem (2-point BVP)**. It is to be distinguished from the **1-point BVP** or **initial value problem (IVP)** considered in Chapter 2, where y and y' are only given at the single point a and where, unlike here, a unique solution can, under very general conditions, always be found. The example in Exercise 4 of the previous section underlines the need to extend our theory.

The conditions (α), (β) are referred to as **homogeneous boundary conditions**, as it is only the ratios $A_1 : B_1$ and $A_2 : B_2$ that matter, not the actual values of A_1, A_2, B_1, B_2.

The function $G = G(x,t)$ occurring in the theorem is called the **Green's function** for the problem consisting of (N) together with the boundary conditions (α), (β). Such Green's functions appear throughout the theory of ordinary and partial differential equations. They are always defined (as below) in terms of solutions of a homogeneous equation and allow a solution of the corresponding non-homogeneous equation to be given (as in Theorem 1 here) in integral form.

Aside On the face of it, the equation (N) would seem to be a special case of the non-homogeneous equation (1) specified at the start of this chapter. In fact, any equation of the form (1) may be written in the form (N) by first 'multiplying it through' by

$$\frac{1}{p_2(x)} \exp\left(\int^x \frac{p_1(t)}{p_2(t)}\, dt\right).$$

The reader should check that the resulting equation is

$$\frac{d}{dx}\left(\exp\left(\int^x \frac{p_1}{p_2}\right)\frac{dy}{dx}\right) + \frac{p_0(x)}{p_2(x)}\exp\left(\int^x \frac{p_1}{p_2}\right) y \;=\; \frac{f(x)}{p_2(x)}\exp\left(\int^x \frac{p_1}{p_2}\right),$$

where each $\int^x$ is the indefinite integral. The equation is now in self-adjoint form and the coefficient functions satisfy the continuity, differentiability and positivity conditions of Theorem 1.

Before proceeding to the proof of the theorem, we shall establish two lemmas. The conditions of the theorem will continue to apply.

Lemma 1 Suppose that y_1 and y_2 are solutions of (H) and that W denotes the Wronskian $W(y_1, y_2)$. Then

$$p(x)W(x) \;=\; A, \quad \text{for each } x \in [a,b],$$

where A is a real constant. If y_1 and y_2 are linearly independent, then A is non-zero.

Proof Since $y_1,\ y_2$ solve (H), we have

$$\begin{aligned} Ly_1 &= (py_1')' + qy_1 = 0, \\ Ly_2 &= (py_2')' + qy_2 = 0. \end{aligned} \tag{15}$$

Hence, on $[a,b]$ we have

$$\begin{aligned} (pW)' &= (p(y_1y_2' - y_2y_1'))' \\ &= (y_1(py_2') - y_2(py_1'))' \\ &= y_1(py_2')' - y_2(py_1')' \\ &= y_1(-qy_2) - y_2(-qy_1), \qquad \text{using (15)}, \\ &= 0. \end{aligned}$$

Thus pW is constant on $[a,b]$. If y_1 and y_2 are linearly independent on $[a,b]$, then Proposition 4 of Chapter 3 tells us that W is never zero in $[a,b]$, and (since we have insisted that p_2 is never zero in $[a,b]$) the proof is complete. □

Note An alternative proof can be based on Exercise 5 of Chapter 3, with

$$p(x) \;=\; \exp\left(\int^x \frac{p_1}{p_2}\right)$$

as in the Aside above.

Lemma 2 There is a solution $y = u$ of $Ly = 0$ satisfying (α) and a solution $y = v$ of $Ly = 0$ satisfying (β) such that u and v are linearly independent over $[a, b]$.

Proof Noting that $Ly = 0$ is the linear equation

$$p(x)y'' + p'(x)y' + q(x)y = 0, \qquad (x \in [a, b])$$

where p, p' and q are continuous on $[a, b]$ and $p(x) > 0$ for each x in $[a, b]$, we see that we may apply the existence part of Theorem 4 of Chapter 2 to find functions u, v solving

$$Lu = 0, \quad u(a) = B_1, \quad u'(a) = -A_1,$$

$$Lv = 0, \quad v(b) = B_2, \quad v'(b) = -A_2.$$

Then, certainly $y = u$ solves $Ly = 0$ taken with (α) and $y = v$ solves $Ly = 0$ taken with (β). Further, u, v must both be not identically zero in $[a, b]$, because of the conditions placed on A_1, B_1, A_2, B_2.

Suppose that C, D are constants such that

$$Cu(x) + Dv(x) = 0, \qquad \text{for each } x \text{ in } [a, b]. \tag{16}$$

As u, v solve $Ly = 0$, they must be differentiable and hence we may deduce

$$Cu'(x) + Dv'(x) = 0, \qquad \text{for each } x \text{ in } [a, b]. \tag{17}$$

Multiplying (16) by A_2 and (17) by B_2, adding and evaluating at $x = b$ gives

$$C(A_2 u(b) + B_2 u'(b)) = 0.$$

If $C \neq 0$, u satisfies (β), as well as $Ly = 0$ and (α), and must be the trivial solution $u \equiv 0$ by one of the hypotheses of the theorem. This contradicts the fact that u is not identically zero in $[a, b]$. Similarly, we can show that

$$D(A_1 v(a) + B_1 v'(a)) = 0$$

and hence, if $D \neq 0$, that $v \equiv 0$ in $[a, b]$, also giving a contradiction. So, C and D must both be zero and u, v must be linearly independent over $[a, b]$. □

Proof of Theorem 1 We first establish uniqueness of the solution. Suppose that y_1, y_2 solve

$$Ly = f(x) \quad \text{together with } (\alpha), (\beta).$$

Then, it is easy to see that $y = y_1 - y_2$ solves

$$Ly = 0 \quad \text{together with } (\alpha), (\beta).$$

By hypothesis, y must be the trivial solution $y \equiv 0$. So, $y_1 = y_2$ and, if a solution to (N) together with (α), (β) exists, it must be unique.

Letting u, v be the functions given by Lemma 2, we can deduce from Lemma 1 that

$$p(x)W(x) = A, \quad \text{for each } x \text{ in } [a,b],$$

where A is a *non-zero* constant and W is the Wronskian $W(u,v)$. We may therefore define $G: [a,b]^2 \to \mathbb{R}$ by

$$G(x,t) = \begin{cases} \dfrac{u(x)v(t)}{A}, & \text{for } a \le x \le t \le b, \\[2ex] \dfrac{u(t)v(x)}{A}, & \text{for } a \le t \le x \le b. \end{cases}$$

Then G is continuous on $[a,b]^2$, as can be seen by letting t tend to x both from above and below. Clearly, G is also twice continuously differentiable when $x < t$ and when $x > t$, and

$$\begin{aligned} \frac{\partial G}{\partial x}(t+0,t) - \frac{\partial G}{\partial x}(t-0,t) &= \lim_{x \to t} \left(\frac{u(t)v'(x)}{A} - \frac{u'(x)v(t)}{A} \right) \\ &= \frac{W(t)}{A} \\ &= \frac{1}{p(t)}, \qquad \text{using Lemma 1,} \end{aligned}$$

for each t in $[a,b]$. It thus remains to prove that, for x in $[a,b]$,

$$y(x) = \int_a^b G(x,t)f(t)\,dt;$$

that is,

$$y(x) = \frac{v(x)}{A}\int_a^x u(t)f(t)\,dt + \frac{u(x)}{A}\int_x^b v(t)f(t)\,dt \tag{18}$$

solves (N) and satisfies the boundary conditions (α), (β). First, note that y is well defined (as G is continuous) and that

$$(19) \qquad y(a) = \frac{u(a)}{A}\int_a^b vf, \qquad y(b) = \frac{v(b)}{A}\int_a^b uf.$$

Using [G] of Chapter 0 to differentiate (18),

$$\begin{aligned} y'(x) &= \frac{v'(x)}{A}\int_a^x uf + \frac{v(x)u(x)f(x)}{A} + \frac{u'(x)}{A}\int_x^b vf - \frac{u(x)v(x)f(x)}{A} \\ &= \frac{v'(x)}{A}\int_a^x uf + \frac{u'(x)}{A}\int_x^b vf, \end{aligned}$$

for each x in $[a, b]$. So,

$$(20) \qquad y'(a) = \frac{u'(a)}{A}\int_a^b vf, \qquad y'(b) = \frac{v'(b)}{A}\int_a^b uf.$$

From (19), (20),

$$A_1 y(a) + B_1 y'(a) = \frac{1}{A}(A_1 u(a) + B_1 u'(a))\int_a^b vf = 0,$$

since u satisfies (α). Thus, y satisfies (α). Similarly, y satisfies (β), using the fact that v satisfies this latter condition. Now,

$$p(x)y'(x) = \frac{p(x)v'(x)}{A}\int_a^x uf + \frac{p(x)u'(x)}{A}\int_x^b vf,$$

and so, differentiating again,

$$\begin{aligned} \frac{d}{dx}(p(x)y'(x)) &= \frac{1}{A}\frac{d}{dx}(p(x)v'(x))\int_a^x uf + \frac{1}{A}p(x)v'(x)u(x)f(x) \\ &\quad + \frac{1}{A}\frac{d}{dx}(p(x)u'(x))\int_x^b vf - \frac{1}{A}p(x)u'(x)v(x)f(x). \end{aligned}$$

Hence, as $pW = p(uv' - u'v) = A$ and $Lu = Lv = 0$,

$$Ly \;=\; \frac{Lv}{A}\int_a^x uf \;+\; \frac{Lu}{A}\int_x^b vf \;+\; f \;=\; f,$$

and our proof is complete. □

Note The Green's function, $G = G(x,t)$, as constructed in the above proof, is independent of the function f on the right-hand side of (N). Put another way: the same G 'works' for every continuous f.

The triviality condition We should like to make three points concerning the theorem's hypotheses that (H), together with (α), (β), only has the identically zero solution.

(i) The condition is used in two places in the proof, in the deduction of Lemma 2 and the proof of uniqueness.

(ii) The condition is by no means always met. The reader should check that

$$y'' + y \;=\; 0, \qquad y(0) \;=\; y(\pi) \;=\; 0$$

does not satisfy it. Variation of parameters is a tool in this case, as a solution of Exercise 4 above will have discovered.

(iii) The condition may be recovered from the linear independence of functions u satisfying $Lu = 0$ together with (α) and v satisfying $Lv = 0$ together with (β); that is, the converse of Lemma 2 is also true, as we shall now show.

Suppose that u satisfies (H) and (α), that v satisfies (H) and (β), and that u, v are linearly independent, so that the Wronskian $W = W(u,v)$ is never zero in $[a,b]$. Further, suppose y satisfies both (α) and (β) as well as (H). Then, by Proposition 6 of Chapter 3, there are constants $c_1,\ c_2$ such that

$$y \;=\; c_1 u + c_2 v$$

and hence

$$y' \;=\; c_1 u' + c_2 v'$$

on $[a,b]$. Therefore,

$$A_1 y(a) + B_1 y'(a) \;=\; c_1(A_1 u(a) + B_1 u'(a)) + c_2(A_1 v(a) + B_1 v'(a))$$

and since both u and y satisfy (α),

$$c_2(A_1 v(a) + B_1 v'(a)) \;=\; 0,$$