By taking the transform of (2a) we mean taking the logarithm of both sides of (2a). In (2f) we are denoting the inverse logarithm, or "antilog," as "$\ln^{-1}$."

$$
\begin{aligned}
z &= (1.7356)(6.8102) && \text{(2a)}\\
\ln z &= \ln\left[(1.7356)(6.8102)\right] \quad \text{[taking the "logarithmic transform" of (2a)]} && \text{(2b)}\\
&= \ln(1.7356) + \ln(6.8102) \quad \text{[by transform property (1)]} && \text{(2c)}\\
&= 0.55135 + 1.91842 \quad \text{[by looking up } \ln(1.7356) \text{ and } \ln(6.8102) \\
&\qquad \text{in the right-hand column of the table]} && \text{(2d)}\\
&= 2.46977 \quad \text{[by addition]} && \text{(2e)}\\
z &= \ln^{-1}(2.46977) \quad \text{[taking the inverse transform]} \\
&= 11.8201 \quad \text{[looking up the inverse in the left- hand column of table].} && \text{(2f)}
\end{aligned}
$$

"Logarithmic Transform" Table

x	$\ln x$
$\vdots$	$\vdots$
1.7356 →	0.55135
$\vdots$	+
6.8102 →	1.91842
$\vdots$	↓
11.8201 ←	2.46977

Figure 4. Logarithmic transform calculation.

Taking the transform of both sides of (2a) moved us to the "transform domain." What was a multiplication in the original domain [i.e., in (2a)] became a simpler operation, addition, in the transform domain [i.e., in (2d)]. We carried out the addition, then returned to the original domain by the inverse transform, the antilog. The steps are indicated in Fig. 4.

For the procedure to work, we needed both the transform and the inverse transform to *exist* and to be *unique*, so we could indeed obtain the transforms in (2d) and the inverse transform in (2f), each in a unique way. These requirements were satisfied because, as we can see from Fig. 3, the logarithm function is a one-to-one function on $0 < x < \infty$.

The Laplace transform, on the other hand, is designed not to facilitate algebraic steps such as multiplication but, primarily, for the solution of linear differential equation initial value problems. Somewhat as the "logarithmic transform" converted the *product* operation in (2a) into the simple *sum* operation in the transform domain, the Laplace transform converts a constant-coefficient linear *differential* equation in the t domain into a linear *algebraic* equation in the transform domain.

Of course, the logarithm is not normally presented, in pre-calculus, in terms of transforms. But, as mentioned above, the foregoing logarithmic transform analogy should help to motivate the material that follows in Section 5.2.

Besides the Laplace transform there is also a **Fourier transform**, which is of comparable importance, but the latter is outside the scope of this text. Fortunately, the transform methodology is so similar, from one transform to another, that an understanding of one is of great help in studying others.

5.2 THE TRANSFORM AND ITS INVERSE

This section is to introduce the transform and to give just enough about the transform and inversion processes so we can get started with differential equation applications in the next section. Additional material on transforms and inverses is given in subsequent sections.

5.2.1 Laplace transform. If a function $f(t)$ defined on $0 \le t < \infty$ is multiplied by e^{-st}, in which s is a constant, and integrated from zero to infinity, the

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result is a function of the parameter s. Known as the **Laplace transform** of $f(t)$, it is denoted as $\mathcal{L}\{f(t)\}$ or as $F(s)$:[1]

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt.} \tag{1}$$

The $\mathcal{L}$ is a script L, for Laplace. Denote the transform of $f(t)$ either as $\mathcal{L}\{f(t)\}$ or as $F(s)$.

In some contexts it is important to allow the parameter s to be a complex number, but in this chapter it will be simpler and sufficient to restrict it to be real.

Thus, for the Laplace transform the input is a function of t and the output, its transform, is a function of s. It is an example of an **integral transform** because it is defined by the *integral* in (1).

The integral is *improper* because of its infinite upper limit, and is defined as the following limit of a sequence of proper integrals (i.e., with finite limits),

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \equiv \lim_{B\to\infty}\int_0^B f(t)e^{-st}\,dt. \tag{2}$$

The definition (2) is in the same spirit as the definition of an infinite series $\sum_0^\infty a_n$ as the limit of the sequence of partial sums, $\lim_{N\to\infty}\sum_0^N a_n$, that one meets in the calculus.

The whole Laplace transform method rests upon the definition (1). As we indicated in Section 5.1, it is designed to convert linear constant-coefficient *differential* equations to simple linear *algebraic* equations, but we won't apply the transform to differential equations until Section 5.3. Here, to get started, let us begin by evaluating the integral (1) for several familiar functions, that is, by working out their Laplace transforms.

EXAMPLE 1. $f(t) = 1$. Evaluate the Laplace transform of the simple function $f(t) = 1$. By the definition of the transform, we have

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty 1e^{-st}\,dt = \lim_{B\to\infty}\int_0^B e^{-st}\,dt \\ &= \lim_{B\to\infty}\left(\frac{e^{-st}}{-s}\Big|_0^B\right) = \lim_{B\to\infty}\left(\frac{1}{s} - \frac{e^{-sB}}{s}\right).\end{aligned} \tag{3}$$

As $B\to\infty$, s remains fixed.

Now take the limit. If $s > 0$, $\lim_{B\to\infty} e^{-sB} = 0$, but if $s < 0$ the limit does not exist (it diverges to infinity). Thus, the right-hand side of (3) is $1/s$ if $s > 0$ and does not exist if $s < 0$. Actually, for the borderline case $s = 0$ equation (3) is not valid, because if $s = 0$, then $\int e^{-st}\,dt = \int dt = t$, not $e^{-st}/(-s)$. Treating that case separately, for $s = 0$ we have $\mathcal{L}\{1\} = \int_0^\infty 1e^0\,dt = \lim_{B\to\infty}\int_0^B dt = \lim_{B\to\infty} B$, which does not exist.

[1]The Laplace transform is named after the great French applied mathematician *Pierre-Simon de Laplace* (1749 – 1827), who contributed chiefly to celestial mechanics but also to fluid mechanics and other branches of science. Although Laplace used integrals of this type in his study of probability, they had already appeared in the work of Euler, and the Laplace transform "method" was developed only later, by the British electrical engineer *Oliver Heaviside* (1850 – 1925). For an historical account see J. L. B. Cooper, "Heaviside and the Operational Calculus," *Math. Gazette*, Vol. 36 (1952), pp. 5–19.

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Remember, the transform of a function $f(t)$ is a function of s; it is *not* a function of t.

Thus,

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}} \tag{4}$$

for $s > 0$. To reiterate, the Laplace transform of $f(t) = 1$ is $1/s$, and it exists for $s > 0$; that is, the domain of definition of $F(s) = 1/s$ is $s > 0$.

COMMENT 1. The transform $F(s)$ of a given function $f(t)$ is a function of s, and any definition of a function includes a statement as to the interval on which it is defined. In this example we found that the s interval on which the transform (4) is defined is $s > 0$, but the s interval of definition differs, in general, from one transform to another. Looking ahead, however, we will not be concerned with what the specific s interval of definition is, as long as there is *some* such interval, an s interval on which the transform integral (1) converges.

COMMENT 2. We examined the borderline case $s = 0$ separately in this example, but the situation at such values of s will not be important and we will ignore them in subsequent examples. ∎

EXAMPLE 2. $f(t) = t^n$. Consider t^n, for any $n = 1, 2, \ldots$, and use integration by parts:

$$\begin{aligned}\mathcal{L}\{t^n\} = \int_0^\infty \underbrace{t^n}_{u} \underbrace{e^{-st}\,dt}_{dv} &= \lim_{B\to\infty}\left(t^n \frac{e^{-st}}{-s}\bigg|_0^B - \int_0^B \frac{e^{-st}}{-s}\, nt^{n-1}\, dt \right) \\ &= \lim_{B\to\infty}\left(\frac{B^n e^{-sB}}{-s}\right) + \frac{n}{s}\mathcal{L}\{t^{n-1}\}. \end{aligned} \tag{5}$$

The limit in (5) does not exist if $s < 0$. If $s > 0$, it is indeterminate because $B^n \to \infty$ and $e^{-sB} \to 0$ as $B \to \infty$. Which one wins? Repeated application of l'Hôpital's rule shows that the limit exists and is zero. For instance, if $n = 2$, then

$$\lim_{B\to\infty} B^2 e^{-sB} = \lim_{B\to\infty} \frac{B^2}{e^{sB}} = \lim_{B\to\infty} \frac{2B}{s\,e^{sB}} = \lim_{B\to\infty} \frac{2}{s^2\, e^{sB}} = 0.$$

Thus,

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}. \tag{6}$$

Putting $n = 1, 2, \ldots$ in (6), in turn, gives

$$\begin{aligned}\mathcal{L}\{t\} &= \frac{1}{s}\mathcal{L}\{t^0\} = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s^2} \quad \text{from (4)}, \\ \mathcal{L}\{t^2\} &= \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\frac{1}{s^2} = \frac{2}{s^3}, \\ \mathcal{L}\{t^3\} &= \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3}{s}\frac{2}{s^3} = \frac{3\cdot 2}{s^4}, \end{aligned}$$

To find the transform of t^5, for instance, we could integrate by parts five times, eventually "knocking down" the t^5 in the integrand to t^0 and thus obtaining a simple integral. But it is more convenient to use the formula (6) recursively, as we've done. The result is given by (7).

and so on. The result is

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad (n = 1, 2, \ldots)} \tag{7}$$

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for $s > 0$. Since $0! = 1$, (7) agrees with (4) for the case $n = 0$, so (7) is valid for $n = 0, 1, 2, \ldots$. ∎

EXAMPLE 3. $f(t) = e^{at}$. Next, consider $f(t) = e^{at}$ in which a is real.

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{at}e^{-st}\,dt = \lim_{B\to\infty}\int_0^B e^{-(s-a)t}\,dt \\ &= \lim_{B\to\infty}\frac{e^{-(s-a)t}}{-(s-a)}\bigg|_0^B = \lim_{B\to\infty}\left(\frac{1}{s-a} - \frac{e^{-(s-a)B}}{s-a}\right).\end{aligned} \tag{8}$$

If $s < a$, the $e^{-(s-a)B} = e^{(a-s)B} \to \infty$ as $B \to \infty$, and the limit does not exist. But, if $s > a$, then $e^{-(s-a)B} \to 0$, the limit exists, and we obtain

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}.} \tag{9}$$

COMMENT. It will also be useful to know the transform of e^{at} if a is complex, say $a = \alpha + i\beta$ in which α and β are real. The result (8) is still valid, but we must re-examine the limit of $e^{-(s-a)B}$ therein as $B \to \infty$. Write

$$e^{-(s-a)B} = e^{-(s-\alpha-i\beta)B} = e^{-(s-\alpha)B}e^{i\beta B}.$$

Since $|e^{i\beta B}| = |\cos\beta B + i\sin\beta B\,| = \sqrt{\cos^2\beta B + \sin^2\beta B} = \sqrt{1} = 1$, everything hinges on the $e^{-(s-\alpha)B}$. The latter diverges to infinity if $s - \alpha < 0$ $(s < \alpha)$, and converges to zero if $s > \alpha$, that is, if $s > \mathrm{Re}\,a$. Thus, (9) holds if $s > \mathrm{Re}\,a$. ∎

Conclusion: If a is real, (9) holds for $s > a$; if a is complex, it holds for $s > \mathrm{Re}\,a$.

5.2.2 Linearity property of the transform.

Since the Laplace transform is defined by an integral, it satisfies the linearity property of integrals, which will help us evaluate transforms just as it helped us evaluate integrals in the calculus:

THEOREM 5.2.1 *Linearity of the Transform*
For any constants α and β,

$$\boxed{\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\},} \tag{10}$$

That is, $\int_0^\infty[\alpha f(t) + \beta g(t)]e^{-st}\,dt = \alpha\int_0^\infty f(t)e^{-st}\,dt + \beta\int_0^\infty g(t)e^{-st}\,dt.$

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for all s for which $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exist.

EXAMPLE 4. Using the Linearity Property. To evaluate the transform of $\sin at$, in which a is real, we could evaluate $\int_0^\infty (\sin at)e^{-st}\,dt$, but it is simpler to use the definition of the sine, together with the transform (9) and the linearity property (10):

The third equality follows from the linearity property (10), and the fourth from (9) with ia in place of "a." In the last two lines we convert $1/(s-ia)$ and $1/(s+ia)$ to the standard Cartesian form "$a+ib$." (Appendix D)

$$\begin{aligned}\mathcal{L}\{\sin at\} &= \mathcal{L}\{\frac{e^{iat}-e^{-iat}}{2i}\} = \mathcal{L}\{\frac{1}{2i}e^{iat} - \frac{1}{2i}e^{-iat}\}\\ &= \frac{1}{2i}\mathcal{L}\{e^{iat}\} - \frac{1}{2i}\mathcal{L}\{e^{-iat}\}\\ &= \frac{1}{2i}\frac{1}{s-ia} - \frac{1}{2i}\frac{1}{s+ia}\\ &= \frac{1}{2i}\frac{1}{s-ia}\frac{s+ia}{s+ia} - \frac{1}{2i}\frac{1}{s+ia}\frac{s-ia}{s-ia}\\ &= \frac{a}{s^2+a^2}.\end{aligned} \tag{11}$$

For what s interval? Recall the margin note for Example 3: $\mathcal{L}\{e^{iat}\} = 1/(s-ia)$ holds for $s > \mathrm{Re}(ia)$, which is zero because ia is purely imaginary, and $\mathcal{L}\{e^{-iat}\}$ holds for $s > \mathrm{Re}(-ia)$, which is also zero. Thus,

$$\boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}} \tag{12}$$

for $s > 0$. ∎

Evaluating $\mathcal{L}\{\cos at\}$ is similar. Begin with

$$\mathcal{L}\{\cos at\} = \mathcal{L}\left\{\frac{e^{iat}+e^{-iat}}{2}\right\} \tag{13}$$

and follow steps analogous to those in (11). The result is

$$\boxed{\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2},} \tag{14}$$

again for $s > 0$. The steps are left for the exercises.

Likewise, we can evaluate the transforms of $\sinh at$ and $\cosh at$ by recalling their definitions $(e^{at}-e^{-at})/2$ and $(e^{at}+e^{-at})/2$, respectively, and using the linearity property and (9); we leave these as exercises as well.

As we continue to evaluate the transforms of various elementary functions, we can begin to generate a *Laplace transform table*. Thus far we have these entries:

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	$f(t)$	$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$
1.	1	$\dfrac{1}{s}$
2.	t^n	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\dfrac{1}{s-a}$
4.	$\sin at$	$\dfrac{a}{s^2+a^2}$
5.	$\cos at$	$\dfrac{s}{s^2+a^2}$

A longer table is given in the endpapers. When we speak of "the table," in this chapter, that is the one we mean.

EXAMPLE 5. Using the Table and the Linearity Property. Evaluate

$$\mathcal{L}\left\{6e^{-2t} + 5\sin 3t\right\}.$$

Life is simple if we find $f(t) = 6e^{-2t} + 5\sin 3t$ in the table (among the end papers). We do not, but we do find e^{at} and $\sin at$, so we can use the linearity property (10) and those two transforms and write

The linearity property (10) also holds for more than two functions; see Exercise 6.

$$\begin{aligned}\mathcal{L}\left\{6e^{-2t} + 5\sin 3t\right\} &= 6\mathcal{L}\left\{e^{-2t}\right\} + 5\mathcal{L}\left\{\sin 3t\right\} \\ &= 6\frac{1}{s+2} + 5\frac{3}{s^2+9}.\end{aligned} \tag{15}$$

For what s interval? The transform $\mathcal{L}\{e^{-2t}\} = 1/(s+2)$ holds for $s > -2$ and $\mathcal{L}\{\sin 3t\} = 3/(s^2+9)$ holds for $s > 0$. *Both* are valid on the overlap of $s > -2$ and $s > 0$, namely, on $s > 0$. Thus, (15) is valid for $s > 0$. ∎

5.2.3 Exponential order, piecewise continuity, and conditions for existence of the transform. We need the transform to *exist* in the first place. For the Laplace transform, the existence of the transform of a given function $f(t)$ defined on $0 \le t < \infty$ amounts to the convergence of its transform integral, for some s interval. Each of the transform integrals of the functions considered thus far has indeed converged for some s interval.

We will now give a set of sufficient conditions on $f(t)$ for its transform to exist, but must first define two concepts, exponential order and piecewise continuity.

Exponential order. A function $f(t)$ is of **exponential order** as $t \to \infty$ if there exist nonnegative constants K, c, and T, such that

$$\boxed{|f(t)| \le Ke^{ct}} \tag{16}$$

If $f(t)$ is real valued, then $|f(t)|$ in (16) means its absolute value; if it is complex valued (as e^{it} is, for instance), then $|f(t)|$ means the modulus of the complex number.

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for all $t \geq T$.[1]

We will call c the **exponential coefficient**. In place of the phrase "for all $t \geq T$" we could say that (16) holds "eventually," that is, for sufficiently large t. *The set of functions of exponential order is the set of functions that do not grow (in magnitude) faster than exponentially*, and that set includes the vast majority of functions of interest to scientists and engineers.

Certainly, every bounded function is of exponential order, because if $|f(t)| \leq M$, say, for all $t \geq 0$, then (16) holds with $K = M$, $c = 0$, and $T = 0$. For instance, $f(t) = 6\cos t$ is of exponential order because $|6\cos t| \leq 6$, so (16) holds with $K = 6$, $c = 0$, and $T = 0$, These values are by no means unique: For $f(t) = 6\cos t$, (16) holds for any $K \geq 6$, for any $c \geq 0$, and for any $T \geq 0$. The numerical values of K, c, T will not be important. Only the existence of such numbers, so that $f(t)$ is of exponential order, will be important.

As one more example, consider $f(t) = t^3$. Showing that $f(t)$ satisfies (16) is equivalent to showing that $|f(t)/e^{ct}| \leq K$ for some $c \geq 0$ and for all sufficiently large T. With $c = 1$, for instance, $t^3/e^t \to 0$ as $t \to \infty$ (by three applications of l'Hôpital's rule), so surely $t^3/e^t \leq 0.01$, for instance, for all sufficiently large t. Thus, t^3 is of exponential order. Similarly, t^N is of exponential order — no matter how large N is.

Piecewise continuity. A function $f(t)$ is **piecewise continuous** on $a \leq t \leq b$ if there exist a finite number of points $t_1, \dots, t_N$ (with $a < t_1 < \cdots < t_N < b$) such that

(i) $f(t)$ is continuous on each open subinterval $a < t < t_1$, $t_1 < t < t_2, \dots,$ $t_N < t < b$, and

(ii) on each subinterval $f(t)$ has finite limits as t approaches the left and right endpoints from within that subinterval.

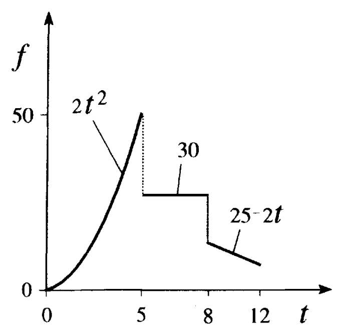

Figure 1. $f(t)$ is piecewise continuous on $0 \leq t \leq 12$.

For instance, consider the function $f(t)$ defined on $0 \leq t \leq 12$, the graph of which is given in Fig. 1. In this case $a = 0$, $b = 12$, $t_1 = 5$, and $t_2 = 8$. The values of $f(t)$ *at* the endpoints $0, 5, 8$, and 12, of the subintervals, are irrelevant insofar as the piecewise continuity of f is concerned; assign any (finite) values you like [such as $f(0) = 837$, $f(5) = 60$, $f(8) = -34$, and $f(12) = \pi$] and $f(t)$ is still piecewise continuous on $0 \leq t \leq 12$. After all, condition (i) involves only the open subintervals, and condition (ii) involves only the limits as the endpoints are approached; neither one involves the values of $f(t)$ *at* the endpoints.

Besides piecewise continuity on a closed interval, we say that $f(t)$ is piecewise continuous **on $0 \leq t < \infty$** *if it is piecewise continuous on $0 \leq t \leq t_0$ for every $t_0 > 0$.*

We now give sufficient conditions on $f(t)$ for its Laplace transform to exist.

[1]For brevity, we will say that $f(t)$ is of exponential order, rather than saying that it is of exponential order "as $t \to \infty$."

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THEOREM 5.2.2 *Existence of the Laplace Transform*
Let $f(t)$ be

(i) *piecewise continuous* on $0 \leq t < \infty$, and

(ii) of *exponential order* as $t \to \infty$, with exponential coefficient c.

Then the Laplace transform of $f(t)$, defined by (1), exists for all $s > c$.

In intuitive language, if $f(t)$ is of exponential order then it grows no faster than Ke^{ct}, for some K and c, so the e^{-st} in $\int_0^\infty f(t)e^{-st}\,dt$ can be made to "clobber" $f(t)$ as $t \to \infty$, ensuring convergence of the integral, by choosing $s > c$.

Proof: Since $f(t)$ is of exponential order, there exist nonnegative constants K, c, and T, such that $|f(t)| \leq Ke^{ct}$ for all $t \geq T$. Using that T, break up the transform integral as

$$\mathcal{L}\{f(t)\} = \int_0^T f(t)e^{-st}\,dt + \int_T^\infty f(t)e^{-st}\,dt \equiv I_1 + I_2, \tag{17}$$

respectively. I_1 exists because it can be written as a finite sum of integrals over intervals on each of which $f(t)e^{-st}$ is continuous, with finite limits at the endpoints. For I_2, use the following comparison test from the calculus: If $|g(t)| \leq h(t)$ for $t \geq a$ and $\int_a^\infty h(t)\,dt$ exists, then $\int_a^\infty g(t)\,dt$ does too. Since $|f(t)| \leq Ke^{ct}$ on $T \leq t < \infty$, by assumption,

$$|f(t)e^{-st}| \leq Ke^{ct}e^{-st} = Ke^{-(s-c)t}. \tag{18}$$

To apply the comparison test to I_2, take "$|g(t)|$" and "$h(t)$" to be the $|f(t)e^{-st}|$ and the $Ke^{-(s-c)t}$ in (18), respectively. Since $\int_T^\infty Ke^{-(s-c)t}\,dt$ exists for all $s > c$, it follows from the comparison test cited above that I_2 does too. Since both I_1 and I_2 exist, it follows from (17) that $\mathcal{L}\{f(t)\}$ exists. ∎

Theorem 5.2.2 is reassuring since the class of functions satisfying conditions (i) and (ii) cover the functions typically encountered in engineering-science applications. Those conditions are sufficient, not necessary. For instance, consider $f(t) = 1/\sqrt{t}$ (Fig. 2). The latter does not satisfy condition (i) because its limit as $t \to 0$ does not exist; thus, $1/\sqrt{t}$ is not piecewise continuous on $0 \leq t < \infty$. Nevertheless, its transform integral does exist and is

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}} \tag{19}$$

for $s > 0$, as is shown in the exercises.

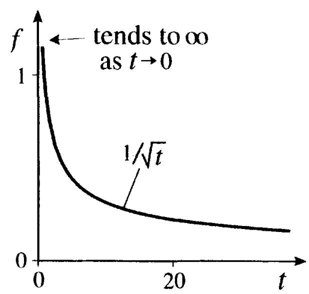

Figure 2. $1/\sqrt{t}$ is *not* piecewise continuous on $0 \leq t < \infty$.

One of the main advantages of the Laplace transform method of solving differential equations is the ease with which it deals with piecewise-continuous forcing functions, as we begin to see in the next example.

EXAMPLE 6. A Piecewise-Defined Function. Consider the piecewise-defined function indicated in Fig. 3. To determine its transform, break up the integral:

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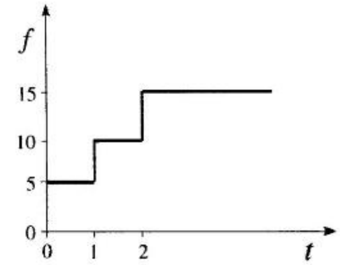

Figure 3. Example of a piecewise-defined function.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty f(t)e^{-st}\,dt = \int_0^1 5e^{-st}\,dt + \int_1^2 10e^{-st}\,dt + \int_2^\infty 15e^{-st}\,dt \\
&= 5\frac{e^{-st}}{-s}\Big|_0^1 + 10\frac{e^{-st}}{-s}\Big|_1^2 + 15\lim_{B\to\infty}\frac{e^{-st}}{-s}\Big|_2^B \\
&= \frac{5}{s}\left(1-e^{-s}\right) + \frac{10}{s}\left(e^{-s}-e^{-2s}\right) - \frac{15}{s}\left(\lim_{B\to\infty} e^{-Bs} - e^{-2s}\right) \\
&= \frac{5}{s}\left(1+e^{-s}+e^{-2s}\right) \quad (s>0).
\end{aligned} \tag{20}$$

COMMENT. This example illustrates how conveniently the Laplace transform handles piecewise-continuous functions: the input $f(t)$ was the "three-tier" piecewise-defined function

$$f(t) = \begin{cases} 5, & 0<t<1, \\ 10, & 1<t<2, \\ 15, & 2<t<\infty, \end{cases} \tag{21}$$

yet the output was the single expression $5\left(1+e^{-s}+e^{-2s}\right)/s$. In fact, piecewise-defined functions will be even simpler to deal with once we introduce the Heaviside step function in Section 5.4. ∎

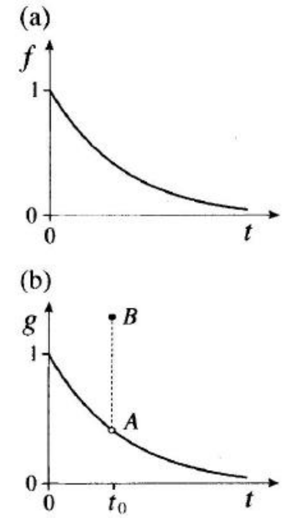

Figure 4. Different functions having the same transform.

5.2.4 Inverse transform. When we use the Laplace transform to solve differential equations in the next section we will need to proceed in both directions: from a given function $f(t)$ to its transform $F(s)$, and from a transform $F(s)$ to the function $f(t)$ "from whence it came." We call the latter the **inverse transform** and denote it as $\mathcal{L}^{-1}\{F(s)\} = f(t)$. Thus, we have the **transform pair**

$$\boxed{\mathcal{L}\{f(t)\} = F(s), \text{ and } \mathcal{L}^{-1}\{F(s)\} = f(t).} \tag{22}$$

For instance, $\mathcal{L}\left\{e^{at}\right\} = 1/(s-a)$ and $\mathcal{L}^{-1}\left\{1/(s-a)\right\} = e^{at}$.

However, the idea fails if inverses are not uniquely determined. That is, if more than one function can have the same transform $F(s)$, then the inverse of $F(s)$ is not defined because it is not *uniquely* defined. In fact, inverses are *not* uniquely defined. To illustrate, let $f(t) = e^{-t}$ (Fig. 4a), and let $g(t)$ be the same as $f(t)$ but with its value at t_0 moved from A to B (Fig. 4b). Surely, the transform of $g(t)$ is the same as that of $f(t)$, namely $1/(s+1)$, because the integrands of their transform integrals differ by a finite value at the point t_0 and therefore have the same areas under their graphs.

But, the good news is twofold: First, such pointwise differences will be of no concern in applications. Second, it turns out that functions satisfying the two conditions of the existence Theorem 5.2.2 and having the same transform can have pointwise differences at most:

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THEOREM 5.2.3 *Uniqueness of the Inverse Transform*
Let $f(t)$ and $g(t)$ satisfy the two conditions in Theorem 5.2.2, so their transforms $F(s)$ and $G(s)$ both exist (on $s > c$ for some constant c). If $F(s) = G(s)$, then $f(t) = g(t)$ for all $t \geq 0$ at which both are continuous.

Note the words "at which both are continuous."

Thus reassured that the idea of the inverse transform is on solid ground, we devote the remainder of this section to the *evaluation* of inverse transforms. That step will rely on the transform table and various helpful properties of the transform and inverse transform.

Linearity of the inverse transform. Recall that the transform is linear:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s). \tag{23}$$

If we express (10) in the reverse direction, we have

$$\boxed{\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t),} \tag{24}$$

Just as the linearity property (23) holds for any finite number of functions (24) does as well.

so the inverse transform is linear too. Both (23) and (24) hold not just for two functions, but for any finite number of them (Exercise 6). The linearity of the inverse will be a key property in evaluating inverse transforms.

5.2.5 Introduction to the determination of inverse transforms. Recall from the calculus that functions that we wish to integrate may not be found in our integral table. But, we learned how to get more mileage out of the table by using various properties of integration such as integration by parts, partial fraction expansions, and substitutions.

Similarly for the evaluation of inverse Laplace transforms, we will use various properties of Laplace inverses, along with various techniques, to get more mileage out of our transform table. Particularly indispensible will be the linearity property of the inverse, stated in (24), and partial fractions. We will illustrate these two in the following examples, just to get started so we can begin to solve differential equations by the Laplace transform method in the next section. Additional useful inversion properties and techniques will be introduced as we go along.

EXAMPLE 7. Using Linearity and the Table. Evaluate $\mathcal{L}^{-1}\left\{\frac{3}{s^5}\right\}$. We could use item 7 of the table, with $n = 4$, but to do that we need a $4!$ in the numerator and we don't have it, because the numerator is 3. So put it there, and compensate by multiplying by $1/4!$. Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{s^5}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{4!}\frac{4!}{s^5}\right\} = \frac{3}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} &= \frac{3}{4!}t^4 \quad \text{(by item 7)}\\ &= \frac{1}{8}t^4.\end{aligned} \tag{25}$$

The second equality in (25) is, by the linearity property (24) with $\beta = 0$: $\mathcal{L}^{-1}\{\alpha F(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\}$.

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COMMENT. More generally, it follows from the linearity of the inverse transform that

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{\alpha}\mathcal{L}^{-1}\{\alpha F(s)\} \tag{26}$$

for any nonzero constant α. That is, *if we would like a (nonzero) constant α introduced into the function of s that is being inverted, just put it there and, to compensate, put $1/\alpha$ out in front.* ∎

EXAMPLE 8. Using Linearity and the Table. Evaluate

$$\mathcal{L}^{-1}\left\{\frac{5}{s+4} + \frac{2}{s^2+9}\right\}.$$

By linearity,

$$\mathcal{L}^{-1}\left\{\frac{5}{s+4} + \frac{2}{s^2+9}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}. \tag{27}$$

The first inverse on the right-hand side is e^{-4t} by item 2 with $a = -4$. For the second we could use item 3, with $a = 3$, but we need an a in the numerator and don't have it. However, we can use (26) and invert as follows:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} == \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin 3t. \quad \text{(by item 3)} \tag{28}$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{5}{s+4} + \frac{2}{s^2+9}\right\} = 5e^{-4t} + \frac{2}{3}\sin 3t. \quad ∎ \tag{29}$$

EXAMPLE 9. Linearity and Item 17. Evaluate

$$\mathcal{L}^{-1}\left\{\frac{6s+3}{(s-5)^4}\right\}. \tag{30}$$

We could use item 17, if not for the s in the numerator. However, we can write that s as $(s-5) + 5$ and proceed as follows:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{6s+3}{(s-5)^4}\right\} &= \mathcal{L}^{-1}\left\{\frac{6(s-5)+33}{(s-5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{(s-5)^3} + \frac{33}{(s-5)^4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{6}{2!}\frac{2!}{(s-5)^3} + \frac{33}{3!}\frac{3!}{(s-5)^4}\right\} = \frac{6}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{(s-5)^3}\right\} \\ &\quad + \frac{33}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s-5)^4}\right\} = 3t^2e^{5t} + \frac{11}{2}t^3e^{5t}.\end{aligned} \tag{31}$$

COMMENT. Note that the simple step $6s + 3 = 33 + 6(s-5)$ is actually a Taylor series expansion of $6s + 3$ about $s = 5$, which series terminates after only two terms. If, instead,

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the numerator in (30) were $6s^2 + 3$, for instance, then its Taylor expansion about 5 would give $153 + 60(s-5) + 6(s-5)^2$. ∎

Using partial fractions. Also of great help will be the technique of partial fractions, as we now illustrate.

EXAMPLE 10. Evaluate

$$\mathcal{L}^{-1}\left\{\frac{7s-5}{s^2-s-2}\right\}. \tag{32}$$

First, factor the denominator: $s^2 - s - 2 = (s-2)(s+1)$. Thus (Appendix A), we can express

$$\frac{7s-5}{s^2-s-2} = \frac{7s-5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}, \tag{33}$$

This is only a modest introductory example. The use of partial fractions to obtain transform inverses will be explained more fully in subsequent sections.

and find that $A = 3$ and $B = 4$. Next,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{7s-5}{s^2-s-2}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{s-2} + \frac{4}{s+1}\right\} \\ &= 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \quad \text{(by linearity)} \\ &= 3e^{2t} + 4e^{-t}. \quad \text{(each by item 2)}\end{aligned} \tag{34}$$

In summary, the form $(7s-5)/(s^2-s+2)$ was not found in the table, so we used partial fractions, then the linearity property (24) and, finally, item 2 in the table.

COMMENT. The technique used in this example works also if the roots for s are complex. However, in this section and in the exercises we consider only examples in which the roots are real. The complex case is also important, and will be included in the next section. ∎

Closure. The Laplace transform of a function $f(t)$ defined on $0 \le t < \infty$ is given by the integral in (1). The integral is improper because of the infinite upper integration limit. For the transform to exist, the integral must converge. If $f(t)$ is piecewise continuous on $0 \le t < t_0$ for every t_0, and $|f(t)|$ does not grow faster than some constant times an exponential e^{ct}, then the integral will indeed converge if s is sufficiently positive. The vast majority of functions likely to arise in applications satisfy those conditions and therefore have Laplace transforms; they are "transformable."

We evaluated the transforms of several elementary functions, constructed a mini table of them, and called attention to the longer table given in the endpapers, upon which we will rely.

We can get more mileage out of the table by using it in conjunction with a number of properties such as the linearity of the transform and inverse transform, along with algebraic techniques such as partial fractions. The same is true in the calculus, in using such properties of integration as linearity and integration by parts to integrate many more functions than are given in the text's integral table.

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Note that the linearity property (23) is equivalent to the reverse statement (24). Linearity is so important that we wrote it both ways. However, as we proceed we will establish many more such properties of the transform, and it seems "overkill" to state each in both directions. Rather, we leave it to the reader to understand that they hold for transforms and also for inverses.

In the next section we begin using the Laplace transform to solve differential equations — which is our purpose in this chapter.

EXERCISES 5.2

NOTE: *We refer to items in the table as "item such and such," and equations in the text are referred to in parentheses. For instance, "(13) and (14)" in Exercise 1, below, means equations (13) and (14) in the text, not items 13 and 14 in the table.*

1. Fill in the steps between (13) and (14).

2. Using the exponential definitions of the hyperbolic sine and cosine, and the integral definition (2), derive these transforms, where a is positive.

(a) $\mathcal{L}\{\sinh at\} = a/(s^2 - a^2)$ for $s > a$
(b) $\mathcal{L}\{\cosh at\} = s/(s^2 - a^2)$ for $s > a$

3. Evaluate the Laplace transform of the given function $f(t)$ by working out the right-hand side of (2), and give the s interval on which the transform exists. HINT: Express sines or cosines in terms of complex exponentials, and hyperbolic sines or cosines in terms of real exponentials. Then, combine those terms with the e^{-st} before integrating.

(a) t	(b) t^2
(c) e^{2t-3}	(d) $5e^{4-3t}$
(e) $\cos(t-2)$	(f) $3\sin(t+5)$
(g) $\sinh(t+1)$	(h) $e^t \cos t$
(i) $e^{-t}\cosh 2t$	(j) $e^{3t}\cos 3t$

4. Piecewise-Defined Functions. Evaluate the transform of the given piecewise-defined function $f(t)$. Is $f(t)$ piecewise continuous on $0 \le t < \infty$?

(a) 0 on $0 \le t < 3$, 50 on $3 \le t < 5$, 0 on $t \ge 5$
(b) 25 on $0 \le t < 5$, 0 on $t \ge 5$
(c) 0 on $0 \le t < 5$, 100 on $t \ge 5$
(d) 50 on $0 < t \le 1$, 25 on $1 < t \le 2$, 0 on $t > 2$
(e) e^t on $0 \le t < 2$, 0 on $t \ge 2$
(f) e^{-t} on $0 \le t < 2$, e^{2-t} on $t \ge 2$
(g) 0 on $0 \le t < 3$, e^{3-t} on $t \ge 3$
(h) t on $0 \le t < 1$, 1 on $t \ge 1$

5. Piecewise Continuous? Let $f(t)$ be the periodic function shown in Fig. 2. Is it piecewise continuous on $0 \le t < \infty$? That is, does it satisfy condition (i) in Theorem 5.2.2? Explain.

6. Linearity. Show that it follows from the linearity property (10) that

(a) $\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t) + \alpha_3 f_3(t)\}$
$\qquad = \alpha_1 F_1(s) + \alpha_2 F_2(s) + \alpha_3 F_3(s)$
(b) $\mathcal{L}\{\alpha_1 f_1(t) + \cdots + \alpha_4 f_k(t)\}$
$\qquad = \alpha_1 F_1(s) + \cdots + \alpha_4 F_4(s)$

In fact, (10) holds not just for two functions but for any finite number of them.

7. Transform, Using Linearity and the Table. Use the linearity property and the Laplace transform table to find the transform of the given function. NOTE: You may use the results stated in Exercise 6.

(a) $6\sin 2t$	(b) $-3e^{-2t}$
(c) $e^t + \cos 4t$	(d) $1 + t^3 e^{-t}$
(e) $\sin t - 3\cos t$	(f) $1 + 7t\cosh 3t$
(g) $t(\sin t + \sinh t)$	(h) $7t^{10} - 4t + 3e^t$
(i) $3(1 - t + t^2)$	(j) $(1-4t)\cos 5t$
(k) $1 + 2t + 3t^2$	(l) $\left(1 - t - 3t^2\right)e^{2t} - 5$
(m) $e^t - e^{2t} - 4t$	(n) $1 + t(1 - e^t - \sin 3t)$

(o) $\cos^2 at$ HINT: $\cos^2 A = (1 - \cos 2A)/2$
(p) $\sin^2 at$ HINT: See part (o).

8. Exponential Order. Show whether or not the given function $f(t)$ is of exponential order. If it is, give any suitable values for K, c, and T.

(a) $5e^{4t}$	(b) $-10e^{-5t}$	(c) $\sinh 2t$
(d) $3 + 2t$	(e) $\sinh t^2$	(f) $e^{4t}\sin t$
(g) $\cos t^3$	(h) $\sin t + 3\cos t$	(i) $(t+1)/(t+2)$
(j) $6t + e^t \cos t$	(k) $4e^t - 5e^{2t}$	(l) $1 + t + t^2$

9. Inversion Using Linearity and the Table. Evaluate the inverse using linearity and one or more of items 1–7 and 17 in the table. Identify any items used.

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(a) $\dfrac{1-5s}{s^3}$ (b) $\dfrac{2}{(s+2)^4}$ (c) $\dfrac{5}{s}-\dfrac{6}{s^4}$
(d) $\dfrac{3s}{2s^2+10}$ (e) $\dfrac{5}{(9s-1)^4}$ (f) $\dfrac{2-5s}{(s+6)^3}$
(g) $\dfrac{1}{(s+1)^5}$ (h) $\dfrac{3}{4-2s}-\dfrac{2}{s^5}$ (i) $\dfrac{6-7s}{2s^2-8}$
(j) $\dfrac{12}{(s-1)^4}$ (k) $\dfrac{2}{s^3}-\dfrac{12}{3-s^2}$ (l) $\dfrac{2s+3}{s^2+6}$
(m) $\dfrac{6s^2+3}{(s-5)^4}$ (n) $\dfrac{10}{s^2-4}$ (o) $\dfrac{5}{2s^2+6}$

10. Using Partial Fractions. By partial fractions, express the given transform in the form $A/(s-a)+B/(s-b)$ and invert.

(a) $\dfrac{1}{s^2-2s-3}$ (b) $\dfrac{8-s}{s^2-s-2}$ (c) $\dfrac{4}{s^2-3s+2}$
(d) $\dfrac{2s}{s^2+4s+3}$ (e) $\dfrac{6}{s(s+6)}$ (f) $\dfrac{6}{3s-s^2}$
(g) $\dfrac{2s+7}{s^2+s-2}$ (h) $\dfrac{7s+8}{3s(s+4)}$ (i) $\dfrac{3s-2}{(2-s)(2+s)}$

11. Multiplication by e^{at}; s-shift. (a) If $\mathcal{L}\{f(t)\}=F(s)$ for $s>c$, and a is any real number, show that

$$\boxed{\mathcal{L}\{e^{at}f(t)\}=F(s-a),} \tag{11.1}$$

which is called the **s-shift formula**.
(b) Use (11.1) and table item 3 to obtain table item 9.
(c) Use (11.1) and table item 4 to obtain table item 10.
(d) Use (11.1) and table item 5 to obtain table item 11.
(e) Use (11.1) and table item 6 to obtain table item 12.

ADDITIONAL EXERCISES

12. The Gamma Function. The transform $\mathcal{L}\{t^n\}$ can be evaluated by integrating $\int_0^\infty t^n e^{-st}\,dt$ by parts n times because by doing so we can "knock down" the t^n to t^0, and $\int_0^\infty t^0 e^{-st}\,dt=\int_0^\infty e^{-st}\,dt$ is readily evaluated. This strategy doesn't work for $\mathcal{L}\{t^p\}$ if p is not an integer, but we *can* evaluate

$$\mathcal{L}\{t^p\}=\int_0^\infty t^p e^{-st}\,dt \qquad (p>-1) \tag{12.1}$$

in terms of the **gamma function** $\Gamma(x)$, which is defined by the formula

$$\boxed{\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt \qquad (x>0).} \tag{12.2}$$

The integral in (12.2) is nonelementary in that it cannot be evaluated in closed form in terms of elementary functions, but it arises often enough in applications for it to have been given a name, the gamma function, and to have been studied extensively. In this exercise we study it, and in the next exercise we use it to evaluate $\mathcal{L}\{t^p\}$ where p is not an integer. To begin, observe that the integral in (12.2) is improper for two reasons: first, the upper limit is ∞ and, second, the integrand is unbounded as $t\to 0$ if $x-1<0$, because $t^{x-1}e^{-t}=t^{x-1}(1-t+t^2/2-\cdots)\sim t^{x-1}$ as $t\to 0$. The more negative the exponent $x-1$, the stronger the "blow-up" of t^{x-1} as $t\to 0$. Nevertheless, it can be shown from the theory of improper integrals that the integral in (12.2) does converge if $x-1$ is not "too" negative, namely, if $x-1>-1$, i.e., if $x>0$. That is why we included the stipulation $x>0$ in (12.2).

(a) Integrating by parts, use (12.2) to show that

$$\boxed{\Gamma(x)=(x-1)\Gamma(x-1) \qquad (x>1).} \tag{12.3}$$

NOTE: The latter *recursion formula* is the most important property of the gamma function. If we know (e.g., by numerical integration) the values of $\Gamma(x)$ on a unit interval such as $0<x\le 1$, then we can use (12.3) to evaluate $\Gamma(x)$ for any $x>1$, by making steps to get into interval on which $\Gamma(x)$ is tabulated. For example,

$$\begin{aligned}\Gamma(3.2)=2.2\Gamma(2.2)&=(2.2)(1.2)\Gamma(1.2)\\&=(2.2)(1.2)(0.2)\Gamma(0.2),\end{aligned}$$

where $\Gamma(0.2)$ is known if $\Gamma(x)$ is indeed known on $0<x\le 1$.

(b) Show, by direct integration, that

$$\Gamma(1)=1. \tag{12.4}$$

(c) Using (12.3) and (12.4), show that if x is a positive integer n, then

$$\boxed{\Gamma(n)=(n-1)!} \tag{12.5}$$

where $0!\equiv 1$.

(d) Besides being able to evaluate $\Gamma(x)$ analytically at $x=1,2,3,\ldots$, we can also evaluate it at $x=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$. In particular, show that

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}. \tag{12.6}$$

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HINT: With $x = 1/2$ in (12.2), change variables from t to u according to $t = u^2$ and show that

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\,du.$$

Then

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty e^{-u^2}\,du \int_0^\infty e^{-v^2}\,dv$$
$$= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\,du\,dv.$$

Regard the latter as a double integral in a Cartesian u, v plane, change from u, v to polar variables r, θ, and remember that in place of the Cartesian area element $du\,dv$ the polar area element is $r dr d\theta$. The resulting double integral should be simpler to evaluate, thanks to the r in the $r dr d\theta$.

(e) Using (12.3) and (12.6), show that

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}. \tag{12.7}$$

NOTE: Similarly, one can evaluate $\Gamma(\frac{7}{2})$, $\Gamma(\frac{9}{2})$, and so on.

13. Finally, the Laplace Transform of t^p. Using (12.2), show that if $p > -1$, then

$$\boxed{\mathcal{L}\left\{t^p\right\} = \frac{\Gamma(p+1)}{s^{p+1}},} \tag{13.1}$$

which is item 8 in the table. For instance, it follows from (12.1) and (12.6) that $\mathcal{L}\left\{t^{-1/2}\right\} = \Gamma(1/2)/s^{1/2} = \sqrt{\pi/s}$.

14. Transforms Tend to Zero as $s \to 0$. Notice that all transforms in our transform table tend to zero as $s \to \infty$. In fact, prove that if $f(t)$ is piecewise continuous on $0 \le t < \infty$ and of exponential order, and its transform is $F(s)$, then

$$\boxed{\lim_{s\to\infty} F(s) = 0.} \tag{14.1}$$

HINT: Recall from the calculus the bound

$$\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt. \tag{14.2}$$

Now, break up the transform integral as we did in (17). Show that $I_1 \to 0$ as $s \to \infty$ by noting that $|f(t)|$ is bounded on $0 \le t \le T$ by some positive number M and using (14.2). Then, show that $I_2 \to 0$ as $s \to \infty$ by using (14.2) and $|f(t)| \le Ke^{ct}$.
NOTE: One can obtain (14.1) more simply as follows:

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty}\int_0^\infty f(t)e^{-st}\,dt$$
$$= \int_0^\infty \lim_{s\to\infty}\left[f(t)e^{-st}\right]dt = \int_0^\infty 0\,dt = 0, \tag{14.3}$$

but that derivation is only formal, not rigorous, because we inverted the order of integration and the limit, in the second equality, without justification.

15. Inverting as a Power Series. If, to invert a given transform, we are willing to end up with a power series rather than a closed-form expression, we can proceed as follows. To illustrate, let us seek the inverse of $F(s) = 1/(s-a)$. Expanding the latter in a Taylor series in s, about $s = 0$, gives

$$F(s) = \frac{1}{s-a} = -\frac{1}{a} - \frac{1}{a^2}s - \frac{1}{a^3}s^2 - \cdots. \tag{15.1}$$

The individual terms are of the simple form s^n, so we might expect to find their inverses in the table, and hence to invert the series term by term. But s^n is not to be found in the table. In fact, as follows from the property (14.1) in Exercise 14, all transforms in our table tend to zero as $s \to \infty$. The positive integer powers of s in (15.1) do not tend to zero as $s \to \infty$, so it appears that they are not invertible, and that (15.1) is not helpful. (More precisely, they are not invertible in terms of functions that satisfy our usual conditions of piecewise continuity and exponential order.) However, suppose we re-express $F(s)$ as

$$F(s) = \frac{1}{s}\frac{1}{1-\dfrac{a}{s}}. \tag{15.2}$$

If we let $a/s \equiv z$, say, then when we expand

$$\frac{1}{1-\dfrac{a}{s}} = \frac{1}{1-z} = 1 + z + z^2 + \cdots$$
$$= 1 + \frac{a}{s} + \frac{a^2}{s^2} + \cdots \tag{15.3}$$

we obtain *inverse* powers of s, which *are* invertible. Thus, (15.2) and (15.3) give

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{a}{s^2} + \frac{a^2}{s^3} + \cdots\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + a\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + a^2\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + \cdots$$
$$= 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots, \tag{15.4}$$

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where, in the second equality, we assumed the infinite series could be inverted term by term. We know the result (15.4) is correct because we know the inverse of $1/(s-a)$ is e^{at} and the final member of (15.4) is indeed the Taylor series of e^{at}. This inversion method is of interest for functions $F(s)$ that are difficult to invert by conventional means, but the drawback is that the inverse obtained is in the form of an infinite series.
The problem: Use this method in each case, obtaining the inverse in the form of a power series; three or four terms of the series will suffice. Then, use the table to invert the given transform in closed form and verify that the power series that you obtained is correct. HINT: In part (a), for instance, factor an s^2 out of the denominator just as we factored an s out of the denominator in (15.2).

(a) $\dfrac{1}{s^2+a^2}$ (b) $\dfrac{1}{s^2-a^2}$

(c) $\dfrac{2s}{(s^2+1)^2}$ (d) $\dfrac{s^2-4}{(s^2+4)^2}$

16. Continuation of Exercise 15. Invert $1/\sqrt{s^2+1}$ using the idea outlined in Exercise 15. That is, write

$$\frac{1}{\sqrt{s^2+1}}=\frac{1}{s\sqrt{1+\dfrac{1}{s^2}}}=\frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}, \tag{16.1}$$

let $1/s^2=z$, say, expand $(1+z)^{-1/2}$ in a Taylor series about $z=0$, replace z by $1/s^2$, invert term by term, and thus show that

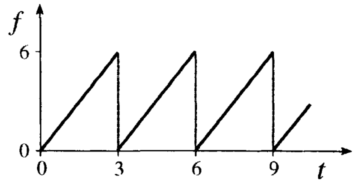
$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\}=1-\frac{1}{4}t^2+\frac{1}{64}t^4-\frac{1}{2304}t^6+\cdots. \tag{16.2}$$

The function defined by the series in (16.2) is the **Bessel function** $J_0(t)$, which will be the subject of Section 6.4.

17. Transform of Periodic Functions. A function $f(t)$ is **periodic with period T** if

$$\boxed{f(t+T)=f(t)} \tag{17.1}$$

for all t's in the domain of t. For instance, the *sawtooth wave*, below, is periodic with period $T=3$, and the segment from $t=0$ to $t=3$, say, is one period. If we repeat that basic unit indefinitely to the right we generate the sawtooth wave.



(a) Let $f(t)$ be periodic, of period T. Show that its transform can be simplified to an integration over only one period, as

$$\boxed{\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-st}}\int_0^T f(t)e^{-st}\,dt.} \tag{17.2}$$

HINT: Write

$$\mathcal{L}\{f(t)\}=\int_0^\infty f(t)e^{-st}\,dt=\int_0^T f(t)e^{-st}\,dt+\int_T^{2T} f(t)e^{-st}\,dt+\cdots. \tag{17.3}$$

Make changes of variables in the integrals so that each has, as new limits, 0 to T, obtaining

$$\mathcal{L}\{f(t)\}=\left(1+e^{-sT}+e^{-2sT}+\cdots\right)\int_0^T f(t)e^{-st}\,dt, \tag{17.4}$$

and use the *geometric series* formula

$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots \qquad (|z|<1) \tag{17.5}$$

to sum the series in (17.4) and obtain (17.2).
(b) For instance, use (17.2) to show that the transform of the sawtooth wave shown above is

$$\mathcal{L}\{f(t)\}=\frac{2}{s^2}-\frac{6}{s}\frac{e^{-3s}}{1-e^{-3s}}. \tag{17.6}$$

(c) For fun, show that the inverse of (17.6) does give us back the sawtooth wave. HINT: This time use the geometric series formula not to sum the series into the closed form $1/(1-z)$, but to expand the $1/(1-e^{-3s})$ into a geometric series in powers of e^{-3s}.

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